Fourier Transform and Spatial Filtering

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Overview

- Last Time:
  - Exercise 1
  - Dithering and Halftoning
  - 2-D signals and systems
- Today
  - 2-D Fourier Transform
  - Image Enhancement via Spatial Filtering
- Additional reference on halftoning:
  - Special Issue on "Digital Halftoning", IEEE Signal Processing Magazine, July 2003 (five articles)

Review of 1-D Fourier Transform

Transform	Time-Domain	Freq.-Domain
Fourier Series (FS)	cont's periodic $x(t)$ $X_n = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x(t) e^{-j2\pi nt/T} dt$ $n \in \mathbb{Z}$	discrete aperiodic (by sinusoids at based harmonic freq.) $x(t) = \sum_{n=-\infty}^{+\infty} X_n \cdot e^{j2\pi nt/T}$
Fourier Transf (FT)	cont's aperiodic $x(t)$ $X(f) = \int_{-\infty}^{+\infty} x(t) e^{-j2\pi ft} dt$ (or in $\Omega = 2\pi f$)	cont's aperiodic $x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(\Omega) e^{j\Omega t} d\Omega$
Discrete-Time Fourier Transf. (DTFT)	discrete aperiodic $x[n]$ $X(\omega) = \sum_{n=-\infty}^{+\infty} x[n] e^{-j\omega n}$	cont's periodic $x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\omega) e^{j\omega n} d\omega$
Discrete Fourier Transf. (DFT)	discrete periodic $x[n]$ $X(k) = \sum_{n=0}^{N-1} x[n] e^{-jnk2\pi/N}$	discrete periodic $x[n] = \frac{1}{N}\sum_{n=0}^{N-1} X(k) \cdot e^{jnk2\pi/N}$

2-D Fourier Transform

- FT for a 2-D continuous function
  - *Horizontal and vertical spatial frequencies (cycles per degree of viewing angle)*

$$F(\zeta_x, \zeta_y) \triangleq FT[f(x,y)] \triangleq \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(x,y)\exp[-j2\pi(x\zeta_x + y\zeta_y)]dxdy,$$

$$f(x,y) \triangleq FT^{-1}[F(\zeta_x, \zeta_y)] \triangleq \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} F(\zeta_x, \zeta_y)\exp[j2\pi(x\zeta_x + y\zeta_y)]d\zeta_x d\zeta_y.$$

  - Separability:
    - *2-D transform can be realized by a succession of 1-D transform along each spatial coordinate*
  - Many other properties can be extended from 1-D FT
    - *convolution in one domain ⇔ multiplication in another domain*
    - *inner product preservation (Parseval energy conservation theorem)*

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$$= \int_{-\infty}^{+\infty}\left[\int_{-\infty}^{+\infty} f(x,y)\exp(-j2\pi x\zeta_x)dx\right]\exp(-j2\pi y\zeta_y)dy,$$

$$f(x,y) \triangleq FT^{-1}[F(\zeta_x,\zeta_y)] \triangleq \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} F(\zeta_x,\zeta_y)\exp[j2\pi(x\zeta_x + y\zeta_y)]d\zeta_x d\zeta_y.$$

  - Separability:
    - *2-D transform can be realized by a succession of 1-D transform along each spatial coordinate*
  - Many other properties can be extended from 1-D FT
    - *convolution in one domain ⇔ multiplication in another domain*
    - *inner product preservation (Parseval energy conservation theorem)*

Freq. Response & Eigen functions for LSI System

- Eigen function of a system
  - Defined as an input function that is reproduced at the output with a possible change only in its amplitude
- Fundamental property of a Linear Shift Invariant System
  - Its eigen functions are
    (recall similar property for 1-D LTI system)
- Frequency response $H(\zeta_x, \zeta_y)$ for a 2-D continuous LSI system is the Fourier Transform of its impulse response

Freq. Response & Eigen functions for LSI System

- Eigen function of a system
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- Fundamental property of a Linear Shift Invariant System
  - Its eigen functions are complex exponentials $\exp[j2\pi(x\,\zeta_x + y\,\zeta_y)]$
    (recall similar property for 1-D LTI system)
- Frequency response $H(\zeta_x, \zeta_y)$ for a 2-D continuous LSI system is the Fourier Transform of its impulse response
  - *represents the (complex) amplitude of the system response for an complex exponential input at spatial frequency (ζ_x, ζ_y)*

$$\exp[j2\pi(x\,\zeta_x + y\,\zeta_y)] \longrightarrow \boxed{H} \longrightarrow H(\zeta_x, \zeta_y)\exp[j2\pi(x\,\zeta_x + y\,\zeta_y)]$$

2-D FT on Discrete 2-D Function

$$X(\omega_1,\omega_2) = \sum_{m,n=-\infty}^{+\infty} x[m,n]\exp[-j(m\omega_1+n\omega_2)]$$

$$x[n_1,n_2] = \frac{1}{(2\pi)^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} X(\omega_1,\omega_2)\, e^{jm\omega_1} e^{jn\omega_2}$$

$$e^{j(\omega_1 m+\omega_2 n)} \longrightarrow \boxed{H} \longrightarrow y[m,n] = H(\omega_1,\omega_2)\, e^{j(\omega_1 m+\omega_2 n)}$$

2-D FT on Discrete/Sampled 2-D Function

$$X(\omega_1, \omega_2) = \sum_{m,n=-\infty}^{+\infty} x[m,n] \exp[-j(m\omega_1 + n\omega_2)]$$

$$= \sum_{n=-\infty}^{+\infty} \left(\sum_{m=-\infty}^{+\infty} x[m,n] e^{-jm\omega_1} \right) e^{-jn\omega_2}$$

$$x[n_1, n_2] = \frac{1}{(2\pi)^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} X(\omega_1, \omega_2)\, e^{jm\omega_1} e^{jn\omega_2}$$

Note: (1) $X(\omega_1, \omega_2)$ is periodic with period 2π in each argument

(2) $\exp\{j(m\omega_1 + n\omega_2)\}$ are eigen functions of 2-D discrete LSI system

$$e^{j(\omega_1 m + \omega_2 n)} \longrightarrow \boxed{H} \longrightarrow y[m,n] = H(\omega_1, \omega_2)\, e^{j(\omega_1 m + \omega_2 n)}$$

2-D DFT (on Discrete Periodic 2-D Function)

$$\begin{cases} Y(k,l) = \dfrac{1}{N} \displaystyle\sum_{m=0}^{N-1} \sum_{n=0}^{N-1} X(m,n) \cdot W_N^{nl} \cdot W_N^{mk} \\ X(m,n) = \dfrac{1}{N} \displaystyle\sum_{k=0}^{N-1} \sum_{l=0}^{N-1} Y(k,l) \cdot W_N^{-nl} \cdot W_N^{-mk} \end{cases}$$

- $W_N = exp\{ - j2\pi / N \}$ complex conjugate of primitive N^{th} root of unity
- Separability: realize 2-D DFT by succession of 1-D DFTs
- Circular convolution in one domain ~ multiplication in another domain

Examples of 2-D DFT

a b

FIGURE 4.3 (a) Image of a 20 × 40 white rectangle on a black background of size 512 × 512 pixels. (b) Centered Fourier spectrum shown after application of the log transformation given in Eq. (3.2-2). Compare with Fig. 4.2.

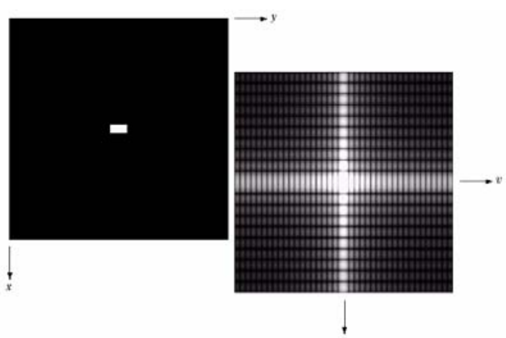
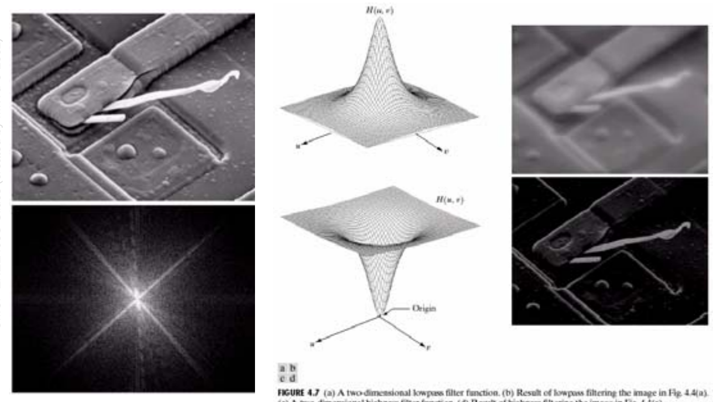


Image examples are from Gonzalez-Woods 2/e online slides.

Frequency Domain View of Linear Spatial Filtering



a b
c d

FIGURE 4.7 (a) A two-dimensional lowpass filter function. (b) Result of lowpass filtering the image in Fig. 4.4(a). (c) A two-dimensional highpass filter function. (d) Result of highpass filtering the image in Fig. 4.4(a).

Image examples are from Gonzalez-Woods 2/e online slides Fig.4.4 & 4.7.

Optical and Modulation Transfer Function

- Optical Transfer Function (OTF) for a LSI imaging system
  - Defined as its normalized frequency response

$$OTF = \frac{H(\xi_x, \xi_y)}{H(0,0)}$$

- Modulation Transfer Function (MTF)
  - Defined as the magnitude of the OTF

$$MTF = |OTF| = \frac{|H(\xi_x, \xi_y)|}{|H(0,0)|}$$

MTF of the Visual System

- Direct MTF measurement of Human Visual System (HVS)
  - Use sinusoidal grating of varying contrast and spatial frequency
    - *The contrast is specified by the ratio of maximum to minimum intensity*
    - *Observation of this grating shows the visibility thresholds at various spatial frequencies*
- Typical MTF has band-pass shape
  - Suggest HVS is most sensitive to mid-freq. and least to high freq.
  - Some variations with viewer & viewing angle

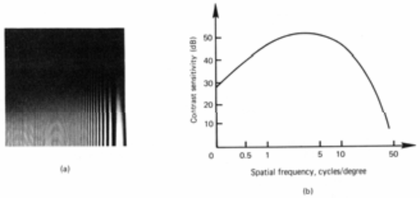


Figure 3.7 MTF of the human visual system. (a) Contrast versus spatial frequency sinusoidal grating; (b) typical MTF plot.

From Jain's book Figure 3.7

Image Enhancement via Spatial Filtering

Spatial Operations with Spatial Mask

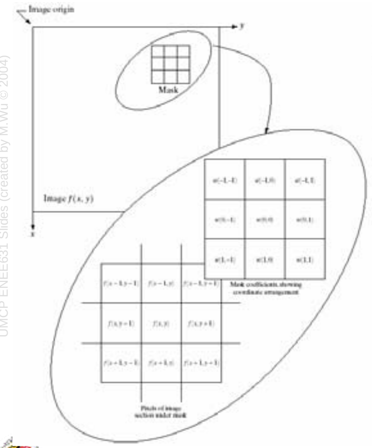


FIGURE 3.32 The mechanics of spatial filtering. The magnified drawing shows a 3 × 3 mask and the image section directly under it; the image section is shown displaced out from under the mask for ease of readability.

- Spatial mask is 2-D finite impulse response (FIR) filter
  - Usually has small support 2x2, 3x3, 5x5, 7x7
  - Convolve this mask with the image
    - *$v(m,n) = \Sigma\, u(m-k, n-l)\, h(k,l)$ … mirror w.r.t. origin, then shift & sum up*
  - Frequency domain interpretation
    - *multiplying DFT(image) with DFT(mask)*

Spatial Averaging Masks

- For softening, noise smoothing, LPF before subsampling(anti-aliasing), etc.
  - "isotropic" (i.e. circularly symmetric / rotation invariant) filters: with response independent of directions

	0	1
0	1/4	1/4
1	1/4	1/4

	-1	0	1
-1	1/9	1/9	1/9
0	1/9	1/9	1/9
1	1/9	1/9	1/9

	-1	0	1
-1	0	1/8	0
0	1/8	1/2	1/8
1	0	1/8	0

Frequency Response of Averaging Mask

Recall: averaging mask is a FIR filter with a square support
=> take FT to get its frequency response: it is a low pass filter

a b
FIGURE 4.3
(a) Image of a 20 × 40 white rectangle on a black background of size 512 × 512 pixels.
(b) Centered Fourier spectrum shown after application of the log transformation given in Eq. (3.2-2). Compare with Fig. 4.2.

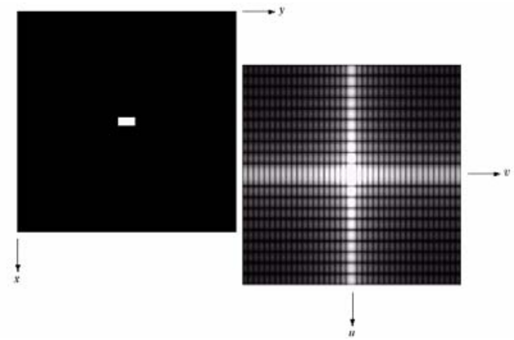


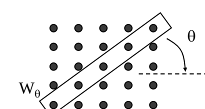
Image examples are from Gonzalez-Woods 2/e online slides.

Suppressing Noise via Spatial Averaging

- Image with iid noise $y(m,n) = x(m,n) + N(m,n)$
- Averaged version
  $v(m,n) = (1/N_w)\ \Sigma\, x(m-k, n-l) + (1/N_w)\ \Sigma\, N(m-k, n-l)$
- Noise variance reduced by a factor of N_w
  - N_w ~ # of pixels in the averaging window
- SNR improved by a factor of Nw if $x(m,n)$ is constant in local window
- Window size is limited to avoid blurring

Directional Smoothing

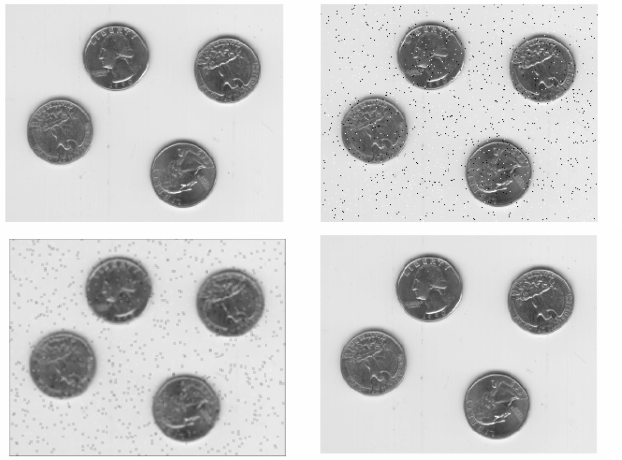
- Problems with simple spatial averaging mask
  - Edges get blurred
- Improvement
  - Restrict smoothing to along edge direction
  - Avoid filtering across edges
- Directional smoothing
  - Compute spatial average along several directions
  - Take the result from the direction giving the smallest changes before & after filtering



- Other solutions
  - Use more explicit edge detection and adapt filtering accordingly

Coping with Salt-and-Pepper Noise

(From Matlab Image Toolbox Guide Fig.10-12 & 10-13)



Averaging Filter

Median Filtering

- Salt-and-Pepper noise
  - Isolated white/black pixels spread randomly over the image
  - Spatial averaging filter may incur blurred output
- Median filtering
  - Take median value over a small window as output ~ nonlinear
    - ◆ $Median\{ x(m) + y(m) \} \neq Median\{x(m)\} + Median\{y(m)\}$
  - Odd window size is commonly used
    - ◆ *3x3, 5x5, 7x7*
    - ◆ *5-pixel "+"-shaped window*
  - Even-sized window ~ take the average of two middle values as output