# On the Equivalence of Evolutionary Stable Strategies

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***Abstract*—Evolutionary game theory (EGT) has been widely adopted in various research fields and recently drawn great attentions in communications and networking. It is considered as an effective tool to analyze how a group of players converges to a stable equilibrium after a period of strategic interactions. Such an equilibrium strategy is defined as the evolutionarily stable strategy (ESS). There are two formal definitions for the ESS, where one of them is a literally mathematical interpretation and the other is a refined Nash equilibrium illustration, and both of them are widely used in the literature. However, the equivalence of these two definitions has not been well investigated. In this letter, we theoretically prove the equivalence of them under some sufficient condition. Moreover, we also show that the sufficient condition is general in communications and networking areas by introducing two examples.**

***Index Terms*—Evolutionary game, evolutionarily stable strategy (ESS), evolutionarily stable state.**

## I. Introduction

EVOLUTIONARY game theory (EGT) studies how a game is repeatedly played by biologically or socially conditioned players who are randomly drawn from large populations [1]. Different from the traditional game theory that focuses on the property of static Nash equilibrium (NE), EGT emphasizes more on the evolutionary dynamics and stability of the whole population's strategies. Recently, it has been widely adopted in communications and networking areas including cognitive radio networks [2], [3], relay networks [4], wireless multicast [5], power control in wireless communication [6], access control [7] and routing [8] in wireless networks.

In EGT, one of the most important concepts is the evolutionarily stable strategy or evolutionarily stable state (ESS), introduced by Maynard and Price in [9]. An ESS is a strategy which, if adopted by a population in a given environment, cannot be invaded by any alternative strategy that is initially rare [1]. Unlike the classical NE which is immune to only one player's strategy deviation, the ESS can prevent a small portion of players choosing alternative strategy, i.e., an equilibrium refinement of the classical NE. Since the concept of stability also plays an important role in communications and networking areas, ESSs in different scenarios have been well investigated in [2]–[8]. Mathematically, there are two formal definitions of the ESS, both of which are widely used. However, to the best of our knowledge, the equivalence of these two definitions has not been well investigated. To the academic integrity of the evolutionary game, in this letter, we study the relationship between these two definitions by theoretically proving the equivalence under some sufficient conditions.

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## II. Problem Formulation

Considering an evolutionary game with a finite strategy set including $M$ pure strategies $\mathcal{A} = \{a_1, a_2, \ldots, a_M\}$, let us define the infinite mixed strategy set as $\mathcal{S} = \{\sigma_i | i = 1, 2, \ldots\}$ and the utility function as $U(\sigma_i, \sigma_j)$, which is the average utility of a player using mixed strategy $\sigma_i$ when meeting with a group of players using mixed strategy $\sigma_j$. Different from the pairwise interaction model where a player only interacts with another player, each player in our model interacts with a group of players, due to which the utility function of every player depends on all players' strategies. According to [1], the first formal definition of the ESS can be written as follows.

*Definition 1:* A strategy $\sigma^\star$ is an ESS if there exists $\bar{\epsilon} > 0$ such that for every strategy $\sigma \neq \sigma^\star$, and for every $0 < \epsilon < \bar{\epsilon}$,

$$U\left(\sigma^\star, \epsilon\sigma + (1-\epsilon)\sigma^\star\right) > U\left(\sigma, \epsilon\sigma + (1-\epsilon)\sigma^\star\right). \quad (1)$$

From the above definition, we can see that for each player, adopting the ESS $\sigma^\star$ is always better than adopting any other strategy, even when a small portion $\epsilon$ of his/her opponents deviates from the ESS $\sigma^\star$. In such a case, if all players adopt the ESS, then no mutant strategy could invade the population even if a small part of players may not be rational and take out-of-equilibrium strategies. Although ***Definition 1*** is the most direct, plain and precise illustration of the ESS, it is not frequently used due to the fact that it is not mathematical solving friendly. On the contrary, the second formal definition of the ESS presented as follows is often used.

*Definition 2:* A strategy $\sigma^\star$ is an ESS if and only if, $\forall\, \sigma \neq \sigma^\star$, $\sigma^\star$ satisfies

$$\begin{aligned} &1)\ U\left(\sigma^\star, \sigma^\star\right) \geq U(\sigma, \sigma^\star), \quad \textit{equilibrium condition}; && (2)\\ &2)\ \text{if } U(\sigma^\star, \sigma^\star) = U(\sigma, \sigma^\star),\\ &\qquad U(\sigma^\star, \sigma) > U(\sigma, \sigma), \quad \textit{stability condition}. && (3) \end{aligned}$$

This definition requires that the ESS $\sigma^\star$ is a best response to itself; and if an alternative best response $\sigma$ exists, then $\sigma^\star$ must be a better response to $\sigma$ than $\sigma$ itself. This definition has the advantage of clear understanding that ESS condition is the combination of NE given by (2), and stability given by (3). Moreover, we can also see that a strict NE can always satisfy this definition as an ESS. Since ***Definition 2*** is relatively easy to mathematically verify and solve the ESS of an evolutionary game, it is widely used in various research fields. Meanwhile, the replicator dynamics are usually used to find the ESS defined by ***Definition 2*** [1], which illustrates the dynamics of the strategy evolution over the whole population. The physical meaning of replicator dynamics is that the higher utility can be obtained by adopting strategy $i$, the probability of adopting this strategy, i.e., the mixed strategy $\sigma_i$, will be higher. In [10], it has been theoretically proved that the asymptotically stable point of the replicator dynamics equation can satisfy ***Definition 2*** as an ESS. In such a case, first defining the replicator dynamics equations and then solving the equations to find asymptotically stable point become a widely adopted procedure for researchers in various research fields to utilize the evolutionary game [2]–[5], which, indeed, are all corresponding to ***Definition 2***.

Up to now, in the literature, there are works using ***Definition 1*** [6]–[8], and also works using ***Definition 2*** [2]–[5]. The question is: "Are these two definitions equivalent under certain condition?". In [1], it has been shown that if the utility function is bilinear, i.e., $U(\sigma_1, \sigma_2)$ is a linear function in terms of both $\sigma_1$ and $\sigma_2$ simultaneously, the equivalence of the two definitions holds. As we know, $U(\sigma_1, \sigma_2)$ can be guaranteed to be linear in terms of $\sigma_1$, since we have

$$U(\sigma_1, \sigma_2) = \sum_{i=1}^{M} \sigma_{1i} U(a_i, \sigma_2) \tag{4}$$

where $\sigma_1 = (\sigma_{11}, \sigma_{12}, \ldots, \sigma_{1M})$. However, under most of circumstances [2]–[8], $U(\sigma_1, \sigma_2)$ is not a linear function in terms of $\sigma_2$. In addition, in [1], it was shown that when the interaction among players is pairwise, i.e., only two players are interacting at one instant, the two definitions are equivalent even when the utility function is not bilinear. This is because, when there is only pairwise interaction, $U(\sigma^\star, \epsilon\sigma + (1-\epsilon)\sigma^\star)$ in ***Definition 1*** can be written as

$$U\left(\sigma^\star, \epsilon\sigma + (1-\epsilon)\sigma^\star\right) = \epsilon U(\sigma^\star, \sigma) + (1-\epsilon) U(\sigma^\star, \sigma^\star). \tag{5}$$

However, the interactions among the population in an evolutionary game may not be pairwise in many scenarios, especially in communications and networking areas where one user's utility is usually affected by all other users' actions. In the sequel, we will give the sufficient conditions which can ensure the equivalence of these two definitions when the utility function of the game is not bilinear restricted and the interactions among the players are not pairwise restricted.

## III. Equivalence of Two Definitions

In this section, we discuss the equivalence of aforementioned two ESS definitions. According to the equilibrium condition of ***Definition 2***, we can see that a strict NE always satisfies ***Definition 2*** as an ESS. However, does a strict NE also always satisfy ***Definition 1*** as an ESS? The following theorem answers this question.

*Theorem 1:* In an evolutionary game whose utility function is continuous in terms of mixed strategy, a strict NE satisfies ***Definition 1*** and ***Definition 2*** simultaneously.

*Proof:* The strict NE satisfying ***Definition 2*** can be easily seen from the equilibrium condition of ***Definition 2***. Therefore, we only focus on the proof that whether a strict NE satisfies ***Definition 1***. The definition of a strict NE can be written as, $\sigma^\star$ is an ESS, if for every strategy $\sigma \neq \sigma^\star$, $\sigma^\star$ satisfies

$$U(\sigma^\star, \sigma^\star) > U(\sigma, \sigma^\star). \tag{6}$$

Since $U(\sigma_i, \sigma_j)$ is continuous at $(\sigma^\star, \sigma^\star)$, according to the Weierstrass definition of continuous functions, we have for any number $\xi_1 > 0$, there exists some number $\delta_1 > 0$, such that for every strategy $\sigma'$ within area $\|\sigma' - \sigma^\star\|_2 < \delta_1$, the value of $U(\sigma^\star, \sigma')$ satisfies that

$$U(\sigma^\star, \sigma^\star) - \xi_1 < U(\sigma^\star, \sigma') < U(\sigma^\star, \sigma^\star) + \xi_1. \tag{7}$$

Note that since the utility function $U(.,.)$ is uniformly continuous on the compact set $[0,1]^M \times [0,1]^M$, $\delta_1$ is independent of $\sigma'$ and $\sigma^\star$ and the same applies for all the following $\delta$s. Similarly, since $U(\sigma_i, \sigma_j)$ is also continuous at $(\sigma, \sigma^\star)$, we have for any number $\xi_2 > 0$, there exists some number $\delta_2 > 0$, such that for every strategy $\sigma'$ within area $\|\sigma' - \sigma^\star\|_2 < \delta_2$, the value of $U(\sigma, \sigma')$ satisfies that:

$$U(\sigma, \sigma^\star) - \xi_2 < U(\sigma, \sigma') < U(\sigma, \sigma^\star) + \xi_2. \tag{8}$$

Since the strict NE ensures that $U(\sigma^\star, \sigma^\star) > U(\sigma, \sigma^\star)$, we can set $\xi_1$ and $\xi_2$ in (7) and (8) as

$$\xi_1 = \xi_2 = \frac{U(\sigma^\star, \sigma^\star) - U(\sigma, \sigma^\star)}{2}. \tag{9}$$

In such a case, according to (7) and (8), there exists some number $\delta_1' > 0$ and $\delta_2' > 0$, for every strategy $\sigma'$ within area

$$\|\sigma' - \sigma^\star\|_2 < \bar{\delta}, \quad \text{where} \quad \bar{\delta} = \min\{\delta_1', \delta_2'\} \tag{10}$$

the values of $U(\sigma^\star, \sigma')$ and $U(\sigma, \sigma')$ satisfy that

$$\frac{U(\sigma^\star, \sigma^\star) + U(\sigma, \sigma^\star)}{2} < U(\sigma^\star, \sigma') < \frac{3U(\sigma^\star, \sigma^\star) - U(\sigma, \sigma^\star)}{2},$$
$$\frac{3U(\sigma, \sigma^\star) - U(\sigma^\star, \sigma^\star)}{2} < U(\sigma, \sigma') < \frac{U(\sigma^\star, \sigma^\star) + U(\sigma, \sigma^\star)}{2}.$$

which indicates that there exists some number $\bar{\delta} > 0$, for every $-\bar{\delta} < \delta < \bar{\delta}$

$$U(\sigma^\star, \sigma^\star + \delta) > U(\sigma, \sigma^\star + \delta) \tag{11}$$

where $\sigma \neq \sigma^\star$ represents all other possible strategies. Thus, there exists some number $\bar{\epsilon} = \min_{\sigma \in [0,1]^M \setminus \sigma^*} (\bar{\delta} / \|\sigma - \sigma^\star\|_2) \geq \bar{\delta}$, for every $0 < \epsilon < \bar{\epsilon}$

$$U\left(\sigma^\star, \sigma^\star + \epsilon(\sigma - \sigma^\star)\right) > U\left(\sigma, \sigma^\star + \epsilon(\sigma - \sigma^\star)\right) \tag{12}$$

which is equivalent with (1) in ***Definition 1***. Thus, we prove that any strict NE always satisfies ***Definition 1***, as well as ***Definition 2***. ■

From ***Theorem 1***, we can see that the strict NE set is a subset of those defined by ***Definition 1*** and ***Definition 2***, as well as the intersection of them. The following theorem shows a sufficient condition that ensures the equivalence of ***Definition 1*** and ***Definition 2*** when the pure strategy set $\mathcal{A}$ is binary, i.e., $(\mathcal{A} = \{a_1, a_2\})$. We have found that in many scenarios [2], [3], [5], [7], the players are only confronted with "do" or "not do", i.e., binary decision. Let us define $(\sigma, 1-\sigma)$ as a player's mixed strategy, which means that he/she adopts strategy $a_1$ with probability $\sigma$ and adopts strategy $a_2$ with probability $1-\sigma$.

*Theorem 2:* In an evolutionary game whose pure strategy set is binary as $\{a_1, a_2\}$, ***Definition 1*** and ***Definition 2*** are equivalent if for all $\sigma \in [0,1]$,

$$f(\sigma) = U(a_1, \sigma) - U(a_2, \sigma) \tag{13}$$

is differentiable and the first order derivative of $f(\sigma)$ is non-positive, i.e., $f'(\sigma) \leq 0, \ \forall \sigma \in [0,1]$.

*Proof:* When the pure strategy set is binary as $\{a_1, a_2\}$, according to (4), we have

$$U(\sigma^\star, \epsilon\sigma+(1-\epsilon)\sigma^\star) = \sigma^\star U(a_1, \epsilon\sigma+(1-\epsilon)\sigma^\star) + (1-\sigma^\star)U(a_2, \epsilon\sigma+(1-\epsilon)\sigma^\star), \quad (14)$$

$$U(\sigma, \epsilon\sigma+(1-\epsilon)\sigma^\star) = \sigma U(a_1, \epsilon\sigma+(1-\epsilon)\sigma^\star) + (1-\sigma)U(a_2, \epsilon\sigma+(1-\epsilon)\sigma^\star). \quad (15)$$

In such a case, ***Definition 1*** can be simplified as, there exists $\bar{\epsilon} > 0$, for every strategy $\sigma \neq \sigma^\star$

$$(\sigma^\star - \sigma)f(\sigma^\star - \epsilon(\sigma^\star - \sigma)) > 0, \qquad \forall\, 0 < \epsilon < \bar{\epsilon}. \quad (16)$$

Similarly, ***Definition 2*** can be simplified as, for every strategy $\sigma \neq \sigma^\star$

$$\begin{cases} (\sigma^\star - \sigma)f(\sigma^\star) > 0, & \text{if } f(\sigma^\star) \neq 0; \\ (\sigma^\star - \sigma)f(\sigma) > 0, & \text{if } f(\sigma^\star) = 0. \end{cases} \quad (17)$$

In addition, by using the mean value theorem, we have for every $\sigma \neq \sigma^\star$ and $0 < \epsilon < \bar{\epsilon}$, there exist one $\sigma'$ satisfying that

$$f(\sigma^* - \epsilon(\sigma^* - \sigma)) - f(\sigma^*) = f'(\sigma')(-\epsilon(\sigma^* - \sigma)). \quad (18)$$

Thus, according to the condition of $f'(\sigma) \leq 0$, we have for every $\sigma \neq \sigma^\star$ and $0 < \epsilon < \bar{\epsilon}$

$$(\sigma^\star - \sigma)f(\sigma^\star - \epsilon(\sigma^\star - \sigma)) \geq (\sigma^\star - \sigma)f(\sigma^\star). \quad (19)$$

To prove the equivalence of between ***Definition 1*** and ***Definition 2***, we consider two cases as follows. In each case, we show that ***Definition 1*** can lead to ***Definition 2***, and vice versa.

- Case 1: $f(\sigma^\star) \neq 0$.

*Definition 1* $\Rightarrow$ *Definition 2:* We prove by contradiction. Suppose there exists a $\sigma' \neq \sigma^\star$ such that

$$(\sigma^\star - \sigma')f(\sigma^\star) \leq 0. \quad (20)$$

Since $\sigma' \neq \sigma^\star$ and $f(\sigma^\star) \neq 0$, we have

$$(\sigma^\star - \sigma')f(\sigma^\star) < 0. \quad (21)$$

According to the continuity property, there exist a sufficiently small $\bar{\epsilon} > 0$, such that $\forall\, 0 < \epsilon < \bar{\epsilon}$

$$(\sigma^\star - \sigma')f(\sigma^\star - \epsilon(\sigma^\star - \sigma')) \leq 0 \quad (22)$$

which contradicts with ***Definition 1*** in (16). Therefore, we have $\forall\, \sigma \neq \sigma^\star$

$$(\sigma^\star - \sigma)f(\sigma^\star) > 0 \quad (23)$$

which is just ***Definition 2***.

*Definition 2* $\Rightarrow$ *Definition 1:* When $f(\sigma^\star) \neq 0$, we have $(\sigma^\star - \sigma)f(\sigma^\star) > 0$ from ***Definition 2***. Thus, according to (19), there exists a $\bar{\epsilon}$ ensuring that for every strategy $\sigma \neq \sigma^\star$, $\forall\, 0 < \epsilon < \bar{\epsilon}$,

$$(\sigma^\star - \sigma)f(\sigma^\star - \epsilon(\sigma^\star - \sigma)) > 0 \quad (24)$$

which is just ***Definition 1***.

- Case 2: $f(\sigma^\star) = 0$.

Since $f'(\sigma) \leq 0$, we have

$$(\sigma^\star - \sigma)(f(\sigma^\star) - f(\sigma)) \leq 0 \Rightarrow (\sigma^\star - \sigma)f(\sigma) \geq 0. \quad (25)$$

Moreover, according to (19), we have

$$(\sigma^\star - \sigma)f(\sigma^\star - \epsilon(\sigma^\star - \sigma)) \geq 0. \quad (26)$$

*Definition 1* $\Rightarrow$ *Definition 2:* We prove by contradiction. Suppose there is a $\sigma' \neq \sigma^\star$ such that

$$(\sigma^\star - \sigma')f(\sigma') = 0 \quad (27)$$

then $f(\sigma') = 0$. Since $f'(\sigma) < 0$, we have $\forall\, \sigma \in [\sigma', \sigma^\star]$ or $\forall\, \sigma \in [\sigma^\star, \sigma']$, $f(\sigma) = 0$. In such a case, there exists a $\bar{\epsilon}$ ensuring that $\forall\, 0 < \epsilon < \bar{\epsilon}$ such that

$$(\sigma^\star - \sigma')f(\sigma^\star - \epsilon(\sigma^\star - \sigma')) = 0 \quad (28)$$

which contradicts with ***Definition 1*** in (16). Therefore, we have $\forall \sigma \neq \sigma^\star$

$$(\sigma^\star - \sigma)f(\sigma^\star) > 0 \quad (29)$$

which is just ***Definition 2***.

*Definition 2* $\Rightarrow$ *Definition 1:* We prove by contradiction. Suppose there is a $\epsilon \in (0, \bar{\epsilon})$ and a $\sigma' \neq \sigma^\star$ such that

$$(\sigma^\star - \sigma')f(\sigma^\star - \epsilon(\sigma^\star - \sigma')) = 0 \quad (30)$$

then we have

$$(\sigma^\star - \epsilon(\sigma^\star - \sigma')) = 0 \quad (31)$$

$$\epsilon(\sigma^\star - \sigma')f(\sigma^\star - \epsilon(\sigma^\star - \sigma')) = 0 \quad (32)$$

which contradicts with ***Definition 2***, where $\forall\, \sigma \neq \sigma^\star$

$$(\sigma^\star - \sigma)f(\sigma) > 0. \quad (33)$$

Thus, $\forall\, 0 < \epsilon < \bar{\epsilon}, \forall\, \sigma \neq \sigma^\star$,

$$(\sigma^\star - \sigma)f(\sigma^\star - \epsilon(\sigma^\star - \sigma)) > 0 \quad (34)$$

which is just ***Definition 1***. ■

***Remark: Theorem 2*** gives a sufficient condition that ensures the equivalence of two ESS definitions when the strategy space is binary. The physical meaning of $f(\sigma)$ is the utility difference between adopting strategy $a_1$ and $a_2$ when other players adopt mixed strategy $\sigma$. Note that $\sigma$ can be interpreted as the portion of players adopting strategy $a_1$ when the population is sufficiently large. Thus, a larger $\sigma$ means the more players adopt strategy $a_1$ in one realization. Therefore, the physical meaning of the sufficient condition $f'(\sigma) \leq 0$ is that the more other players adopt strategy $a_1$, one player's utility difference between adopting $a_1$ and $a_2$ is smaller. One special case is that $U(a_1, \sigma)$ is a decreasing function in $\sigma$ and $U(a_2, \sigma)$ is an increasing function in $\sigma$ (a decreasing function in $1 - \sigma$). This is corresponding to the concept of negative network externality, where the more users adopt the same strategy, the utility obtained by each individual user is less. Such a negative externality phenomenon is quite common in communications and networking areas [11].

As aforementioned, by solving the replicator dynamics equations, the ESS that satisfies ***Definition 2*** can be found, which is a common approach to find ESS in the existing literature. In EGT, when the dimension of strategy space increases, the number of replicator dynamics equations needed to characterize the ESS increases, due to which finding the ESS becomes much more challenging. Specifically, with $M$ possible strategies in the strategy space, we need $M-1$ replicator dynamics equations to characterize the ESS. Moreover, since the $M-1$ equations are generally coupled together, we need to solve them jointly to find the ESS, i.e., the complexity order of finding ESS is linear in terms of the dimension of strategy space, i.e., $o(M)$, which is very challenging. Therefore, in the literature, binary strategy is often used instead.

## IV. Examples in Communications and Networking

In this section, we will show that the sufficient condition in ***Theorem 2*** is general by introducing two examples, where ***Definition 2*** is adopted in the first example [2] while ***Definition 1*** is used in the second example [7]. By showing the equivalence, the conclusions drawn in the two examples can be applicable to both definitions, i.e., the ESS derived in the [2] can also prevent a small portion of players' deviation, while the ESS derived in [7] can also ensure the stability condition.

### A. Cooperative Spectrum Sensing

In cognitive radio networks, secondary users (SUs) cooperatively sense the primary channel state to enhance the detection probability. In [2], the author proposed an evolutionary cooperative spectrum sensing game, where a binary strategy set is considered: $\mathcal{A}=\{C,D\}$ including "$C$" representing contributing to spectrum sensing and "$D$" representing denying to contribute. In this cooperative spectrum sensing game, contributing to spectrum sensing would incur extra energy and time consumption, while if no one contributes, all of them are unable to access the primary channel since the channel state cannot be obtained. In [2], the author derived the ESSs according to ***Definition 2*** through analyzing the utility difference between adopting "$C$" and "$D$" as follows:

$$U(C,\sigma)-U(D,\sigma)=\frac{U_0}{K}\left[\frac{\tau(1-\sigma)^K-\tau}{\sigma}+K(1-\sigma)^{K-1}\right] \quad (35)$$

where $\sigma$ denotes the probability of one SU adopting strategy "$C$", $U_0$ denotes the throughput achieved by a free rider who relies on the contributors' sensing outcomes, $K$ is the number of primary channels and $\tau$ denotes the fraction of the sensing time over the duration of a time slot. From (35), we can see that $f(\sigma)=U(C,\sigma)-U(D,\sigma)$ is a decreasing function in terms of $\sigma$, which satisfies the sufficient condition in ***Theorem 2***. Therefore, in this evolutionary cooperative spectrum sensing game, the ESSs defined by both definitions are equivalent.

### B. Wireless Network Access

In a wireless network, since the network resources are limited in terms of time, space and frequency domains, a common phenomenon is that the more user access the network, the less throughput can be obtained by each user, i.e., the characteristic of negative network externality. In [7], the author formulated the wireless network access problem using evolutionary game and considered a binary strategy set: $\mathcal{A}=\{T,S\}$, where "$T$" means to transmit and "$S$" means to stay quiet. In this game, if all users choose to transmit packets, the collision probability would be extremely high; while all users staying quiet would lead to the loss of potential reward from successful transmission. The utility function of adopting strategy "$T$" and "$S$" were summarized in [7] as follows:

$$U(T,\sigma)=\mu(V+\Delta)\sum_{k\geq 0}(1-\sigma)^k\mathbb{P}(K=k)-\mu(\Delta+\delta), \quad (36)$$

$$U(S,\sigma)=-\mu\kappa\sum_{k\geq 0}(1-\sigma)^k\mathbb{P}(K=k), \quad (37)$$

where $\sigma$ denotes the probability of one user adopting strategy $T$, $\mu$ denotes the probability that a user has its receiver within its range, $\delta$ is the cost of transmitting one packet, $\Delta$ is the cost of packet collision, $V$ is the reward from successful packet transmission, $\kappa$ is the regret cost if all users stay quiet and $\mathbb{P}(K=k)$ is the probability of one user with $k$ other interfering users. In [7], the author analyzed the ESSs of this game according to ***Definition 1***. From (36) and (37), we can see that $U(T,\sigma)$ is a decreasing function in terms of $\sigma$ and $U(S,\sigma)$ is an increasing function in terms of $\sigma$ (decreasing function in terms of $1-\sigma$), i.e., the negative network externality, which is a special case of the sufficient condition in ***Theorem 2***. Therefore, in this evolutionary wireless network access game, the ESSs defined by both definitions are equivalent.

## V. Conclusion

In this letter, we discussed two definitions of the evolutionarily stable strategy and the equivalence of them. We first showed that a strict NE can satisfy both definitions. Then, we provided a sufficient condition that ensures the two definitions are equivalent when the pure strategy set is binary, which is shown to be quite general in communications and networking areas by discussing two examples.

## References

[1] J. Hofbauer and K. Sigmund, *Evolutionary Games and Population Dynamics*. Cambridge, U.K.: Cambridge Univ. Press, 1988.

[2] B. Wang, K. J. R. Liu, and T. C. Clancy, "Evolutionary cooperative spectrum sensing game: How to collaborate?" *IEEE Trans. Commun.*, vol. 58, no. 3, pp. 890–900, Mar. 2010.

[3] C. Jiang, Y. Chen, Y. Gao, and K. J. R. Liu, "Joint spectrum sensing and access evolutionary game in cognitive radio networks," *IEEE Trans. Wireless Commun.*, vol. 12, no. 5, pp. 2470–2483, May 2013.

[4] Z. Zhang and H. Zhang, "A variable-population evolutionary game model for resource allocation in cooperative cognitive relay networks," *IEEE Commun. Lett.*, vol. 17, no. 2, pp. 361–364, Feb. 2013.

[5] B. Hu, H. V. Zhao, and H. Jiang, "Incentive mechanism in wireless multicast," in *Proc. IEEE ICASSP*, 2011, pp. 2344–2347.

[6] H. Tembine, E. Altman, R. El-Azouzi, and W. H. Sandholm, "Evolutionary game dynamics with migration for hybrid power control in wireless communications," in *Proc. IEEE CDC*, 2008, pp. 4479–4484.

[7] H. Tembine, E. Altman, R. El-Azouzi, and Y. Hayel, "Evolutionary games in wireless networks," *IEEE Trans. Syst., Man, Cybern. B, Cybern.*, vol. 40, no. 3, pp. 634–646, Jun. 2010.

[8] M. P. Anastasopoulos, P.-D. M. Arapoglou, R. Kannan, and P. G. Cottis, "Adaptive routing strategies in IEEE 802.16 multi-hop wireless backhaul networks based on evolutionary game theory," *IEEE J. Sel. Areas Commun.*, vol. 26, no. 7, pp. 1218–1225, Sep. 2008.

[9] J. M. Smith and G. R. Price, "The logic of animal conflict," *Nature*, vol. 246, no. 5427, pp. 15–18, Nov. 1973.

[10] I. M. Bomze and E. E. van Damme, "A dynamical characterization of evolutionarily stable states," *Ann. Oper. Res.*, vol. 37, no. 1, pp. 229–244, 1992.

[11] C. Jiang, Y. Chen, and K. J. R. Liu, "Multi-channel sensing and access game: Bayesian social learning with negative network externality," *IEEE Trans. Wireless Commun.*, vol. 13, no. 14, pp. 2176–2188, Apr. 2014.