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NP-hardness of the stable matrix in unit interval family problem in discrete time

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Abstract

We show that to determine if a family of matrices, each with parameters in the unit interval, contains a matrix with all eigenvalues inside the unit circle is an NP-hard problem. We also discuss how this problem is closely related to the widespread problem of power control in wireless systems. © 2001 Elsevier Science B.V. All rights reserved.

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1. Introduction

In a wireless network, an optimal solution to the automatic power control problem can be obtained using Frobenius' theorem [10]. Once a signal to interference and noise ratio (SINR) has been assigned to each user, the existence of an optimal power control solution requires that the pathgain matrix should have all its eigenvalues inside the unit circle [8]. If the pathgain matrix conforms with this constraint, then the SINR values are called *feasible*. In a wireless system where users have different SINR requirements due to different multimedia service types, the system designer endeavors to assign the highest possible SINR levels to all the users, according to their service types. The problem of finding a set of feasible SINR levels that are somehow optimal, in the sense that they are as high as possible considering the priorities of each service type, has been proposed in [6]. This problem is closely related to the "Stable Rank One Perturbed Matrix" problem [1], except that it refers to discrete-time stability and not continuous-time stability. Henceforth, we denote the discrete-time problem as *Power-P*. One method to show that Power-P is NP-hard is by using the problem "stable matrix in unit interval family" problem [1] for discrete time, hereafter denoted as *SMIUIF*.

The principal undertaking of this paper is to show that SMIUIF is NP-hard. This problem, in turn, can be used to show NP-hardness for some communications problems, as discussed in Section 3, including the problem mentioned above, Power-P.

We shall denote the set of positive integers as $\mathbb{Z}^+$, and the set of rational numbers as $\mathbb{Q}$. The problem statement is:

SMIUIF. *Let $n \in \mathbb{Z}^+$, and let I be a set of coordinates, $I = \{(i,j)\colon 1 \leqslant i,j \leqslant n\}$, which is partitioned into two subsets, I_1, and I_2. Let $\mathscr{A}$ be a set of real matrices, defined by $\mathscr{A} = \{A\colon A(i,j) \in \mathbb{Q}$, for $(i,j) \in I_1$ and $A(i,j) \in [-1,1]$, for $(i,j) \in I_2\}$. Does $\mathscr{A}$ contain a matrix with all of its eigenvalues inside the unit circle?*

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From a linear system's perspective, our work supplements that which was done in [7] where NP-hardness is shown for several robust stability problems. The second problem addressed in [7] is to determine if the members of an interval family, similar to our $\mathscr{A}$, have their eigenvalues inside or on the unit circle. The difference between that problem and ours is that the problem in [7] asks if all members of the family are stable, we ask if there is at least one member which is stable.

SMIUIF has been shown to be NP-hard for the stabilization in a continuous-time linear feedback system by Blondel and Tsitsiklis [1], but not for the discrete-time linear feedback system. For the continuous-time case, the matrix is stable if the eigenvalues are on the left side of the complex plane. For the discrete-time case, which we address, the matrix is stable if the eigenvalues are inside of the unit circle. The proof used here is similar to that used in [1], except that the family of matrices used in the instance of SMIUIF, $\mathscr{A}$, must be defined differently, and the parameters used in describing the members of $\mathscr{A}$ must be chosen differently.

The distinction between continuous-time and discrete-time cases is noteworthy in that the latter is relevant to some communications' problems. NP-hardness for the discrete-time case has been assumed to be true in the control of systems whose components are distributed over a network, where the network has a limited communication capacity [3,4]. However, it was not proven. As mentioned before, the optimal power control in wireless systems is another example. This is discussed further in Section 3.

2. Proof of NP-hardness for SMIUIF

To show that SMIUIF is NP-hard, we use the known NP-complete problem [2]:

Problem 1. *Let $\{a_i\}_{i=1,\dots,l}$ be a set of integers. Do there exist $t_1,\dots,t_l \in \{-1,1\}$ such that*

$$\sum_{i=1}^{l} a_i t_i = 0\,? \tag{1}$$

Here, three intermediate results are presented, in order to streamline the proof for the problem of interest.

Lemma 1. *Let $\boldsymbol{a}$ be a real vector of length l, and k be a real scalar. The $(l+1)\times(l+1)$ size matrix*

$$A_1 = \begin{pmatrix} & & & 0 \\ & -k\boldsymbol{a}\boldsymbol{a}^{\mathrm{T}} & & \vdots \\ & & & 0 \\ 0 & \cdots & 0 & 0 \end{pmatrix}$$

has eigenvalues zero with multiplicity l and a simple eigenvalue $-k\boldsymbol{a}^{\mathrm{T}}\boldsymbol{a}$.

Proof. A_1 has rank 1, and the rest follows. □

Lemma 2. *The $(l+1)\times(l+1)$ matrix*

$$A_2 = \begin{pmatrix} 0 & \cdots & 0 & -\alpha x_1 \\ \vdots & & \vdots & \vdots \\ 0 & \cdots & 0 & -\alpha x_l \\ \alpha x_1 & \cdots & \alpha x_l & k\beta \end{pmatrix}$$

where $\alpha = 3/(4\sqrt{l})$, $k = 1/(2\boldsymbol{a}^{\mathrm{T}}\boldsymbol{a})$, $\beta = 5\boldsymbol{a}^{\mathrm{T}}\boldsymbol{a}/2$, and $x_i \in \{-1,1\}$, for $i=1,\dots,l$, has all eigenvalues inside the unit circle.

Proof. We can show that the characteristic polynomial of A_2 is

$$\begin{aligned}\det(\lambda I - A_2) &= (\lambda - k\beta)\lambda^l + l\alpha^2\lambda^{l-1} \\ &= \lambda^{l-1}(\lambda^2 - k\beta\lambda + l\alpha^2).\end{aligned} \tag{2}$$

Eq. (2) has $l-1$ roots at zero, and two additional roots at

$$\lambda = \frac{k\beta \pm \sqrt{k^2\beta^2 - 4l\alpha^2}}{2}. \tag{3}$$

From the premise, we know that $k\beta < 2\alpha\sqrt{l}$, which makes the discriminant negative, so that $|\lambda|^2 = l\alpha^2$. Also from the premise, it is known that $\alpha < 1/\sqrt{l}$. Thus all eigenvalues of A_2 are inside the unit circle. □

Lemma 3. *Let $B(\theta)$ be the $(l+1)\times(l+1)$ matrix defined by*

$$B(\theta) = \begin{pmatrix} -(I_{l\times l} + k\boldsymbol{a}\boldsymbol{a}^{\mathrm{T}}) & \theta\alpha\boldsymbol{y}_0 \\ \theta\alpha\boldsymbol{x}_0^{\mathrm{T}} & k\beta - 1 \end{pmatrix},$$

where $\boldsymbol{x}_0$ and $\boldsymbol{y}_0$ are vectors in $[-1,1]^l$, $\boldsymbol{a}$ is a vector in $\mathbb{Z}^l$, $\theta \in [0,1]$, $\alpha \in (1/(2\sqrt{l}), 1/\sqrt{l})$, and $k\beta \in (1, 2\alpha\sqrt{l})$. If $B(\theta)$ has a zero eigenvalue, then

$$(\alpha\theta)^2\boldsymbol{x}_0^{\mathrm{T}}\left(I - \frac{k}{k\boldsymbol{a}^{\mathrm{T}}\boldsymbol{a}+1}\boldsymbol{a}\boldsymbol{a}^{\mathrm{T}}\right)\boldsymbol{y}_0 = -(k\beta - 1). \tag{4}$$

Proof. By premise, $\lambda_i = 0$ for some eigenvalues of $B(\theta)$. Denote the eigenvector associated to λ_i as $\boldsymbol{v}_i = \binom{\boldsymbol{\Gamma}}{q}$, where $\boldsymbol{\Gamma}$ is a vector and q is a scalar. So,

$$B(\theta)\boldsymbol{v}_i = \boldsymbol{0} \Rightarrow \begin{cases} -(I + k\boldsymbol{a}\boldsymbol{a}^{\mathrm{T}})\boldsymbol{\Gamma} + \theta\alpha q\boldsymbol{y}_0 = \boldsymbol{0} \\ \theta\alpha\boldsymbol{x}_0^{\mathrm{T}}\boldsymbol{\Gamma} + (k\beta - 1)q = 0. \end{cases}$$

Now, $-(I + k\boldsymbol{a}\boldsymbol{a}^{\mathrm{T}})$ is negative definite, so it is invertible, therefore,

$$\boldsymbol{\Gamma} = \theta\alpha q(I + k\boldsymbol{a}\boldsymbol{a}^{\mathrm{T}})^{-1}\boldsymbol{y}_0.$$

Since $q = 0$ requires that $\boldsymbol{v}_i = \boldsymbol{0}$, it can be deduced that $q \neq 0$, which results in

$$(\alpha\theta)^2\boldsymbol{x}_0^{\mathrm{T}}(I + k\boldsymbol{a}\boldsymbol{a}^{\mathrm{T}})^{-1}\boldsymbol{y}_0 + (k\beta - 1) = 0.$$

Using the matrix inversion lemma [9], Eq. (4) is finally obtained. □

Theorem 1. *Problem* SMIUIF *is NP-hard.*

Proof. The proof has three parts [5]:

Part 1: An instance of Problem 1 is reduced to an instance of Problem SMIUIF.

Let the integers a_i, $i = 1,\dots,l$, be an instance of Problem 1. Let $\boldsymbol{x},\boldsymbol{y} \in [-1,1]^l$, and $\mathscr{A}$ be a family of matrices which are parameterized by $\boldsymbol{x}$ and $\boldsymbol{y}$:

$$\mathscr{A} = \{A(\boldsymbol{x},\boldsymbol{y}) : \boldsymbol{x},\boldsymbol{y} \in [-1,1]^l\},$$

where

$$A(\boldsymbol{x},\boldsymbol{y}) = \begin{pmatrix} -k\boldsymbol{a}\boldsymbol{a}^{\mathrm{T}} & \alpha\boldsymbol{y} \\ \alpha\boldsymbol{x}^{\mathrm{T}} & k\beta \end{pmatrix}, \tag{5}$$

with $\alpha = 3/(4\sqrt{l})$, $k = 1/(2\boldsymbol{a}^{\mathrm{T}}\boldsymbol{a})$, $\beta = 5\boldsymbol{a}^{\mathrm{T}}\boldsymbol{a}/2$. The family of matrices $\mathscr{A}$ is an instance of Problem SMIUIF.

Part 2: If the instance of Problem 1 is true, then the instance of Problem 2 is true. In other words, if there exists a set $\{t_i\}_{i=1,\dots,l}$, that satisfies Eq. (1), then there is at least one stable matrix in the interval family, $\mathscr{A}$.

Let the collection of $t_i \in \{-1,1\}$, $i = 1,\dots,l$, satisfy (1). Consider the member of $\mathscr{A}$ which has $\boldsymbol{x}^{\mathrm{T}} = (t_1,\dots,t_l)$, denoted $\boldsymbol{x}_0$, and $\boldsymbol{y} = -\boldsymbol{x}_0$, denoted $\boldsymbol{y}_0$. Note that, because of Eq. (1), $\boldsymbol{x}_0^{\mathrm{T}}\boldsymbol{a} = \boldsymbol{a}^{\mathrm{T}}\boldsymbol{x}_0 = 0$. We show that this matrix, $A_0 = A(\boldsymbol{x}_0,\boldsymbol{y}_0)$, is verified to be stable in polynomial time, i.e., it has all of its eigenvalues inside the unit circle. Indeed, A_0 can be decomposed as follows:

$$A_0 = \underbrace{\begin{pmatrix} -k\boldsymbol{a}\boldsymbol{a}^{\mathrm{T}} & 0 \\ & \vdots \\ 0\dots & 0 \end{pmatrix}}_{\text{denote } A_1} + \underbrace{\begin{pmatrix} 0 & \dots & 0 & \\ \vdots & & \vdots & -\alpha\boldsymbol{x}_0 \\ 0 & \dots & 0 & \\ & \alpha\boldsymbol{x}_0^{\mathrm{T}} & & k\beta \end{pmatrix}}_{\text{denote } A_2}.$$

Lemma 1 states that matrix A_1 has a zero eigenvalue with multiplicity l, and a simple eigenvalue at $-k\boldsymbol{a}^{\mathrm{T}}\boldsymbol{a}$, which, by premise, falls inside the interval $0 < k\boldsymbol{a}^{\mathrm{T}}\boldsymbol{a} < 1$. Lemma 2 states that matrix A_2 has all its eigenvalues inside the unit circle.

Let λ be an eigenvalue of A_0, with eigenvector $\boldsymbol{v}$:

$$A_0\boldsymbol{v} = (A_1 + A_2)\boldsymbol{v} = \lambda\boldsymbol{v} \quad\Rightarrow\quad (A_1^2 + A_1A_2)\boldsymbol{v} = \lambda A_1\boldsymbol{v}. \tag{6}$$

The fact that $\boldsymbol{x}_0^{\mathrm{T}}\boldsymbol{a} = \boldsymbol{a}^{\mathrm{T}}\boldsymbol{x}_0 = 0$, results in $A_1A_2 = A_2A_1 = 0$, so

$$A_1(A_1\boldsymbol{v}) = \lambda(A_1\boldsymbol{v}). \tag{7}$$

Therefore, if $(A_1\boldsymbol{v})$ is not zero, then λ is an eigenvalue of A_1. Conversely, if $\boldsymbol{v}$ is in the nullspace of A_1, then λ is an eigenvalue of A_2. Therefore, every eigenvalue of A_0 is also an eigenvalue of A_1 or A_2. Since both A_1 and A_2 are stable, then so is A_0, and every result used for this conclusion is obtained in polynomial time.

Part 3: If the instance of Problem 2 is true, then the instance of Problem 1 is true. That is, if there is a stable matrix in the interval family, $\mathscr{A}$, then there exists a solution, $\{t_i\}_{i=1,\dots,l}$, that satisfies Eq. (1).

Let $\mathscr{A}$ contain a stable matrix and let $\tilde{\boldsymbol{x}},\tilde{\boldsymbol{y}} \in [-1,1]^l$ be the two vector parameters of that stable matrix, so that $A(\tilde{\boldsymbol{x}},\tilde{\boldsymbol{y}})$, as given by (5), has all of its eigenvalues inside the unit circle.

Consider the parameterized family of matrices:

$$B(\theta) = \begin{pmatrix} -(I_{l\times l} + k\boldsymbol{a}\boldsymbol{a}^{\mathrm{T}}) & \theta\alpha\tilde{\boldsymbol{y}} \\ \theta\alpha\tilde{\boldsymbol{x}}^{\mathrm{T}} & k\beta - 1 \end{pmatrix},$$

where $\theta \in [0,1]$. The behavior of the eigenvalues of $B(\theta)$ is studied as θ varies.

When $\theta = 0$,

$$B(0) = \begin{pmatrix} -(I_{l\times l} + k\boldsymbol{a}\boldsymbol{a}^{\mathrm{T}}) & \boldsymbol{0} \\ \boldsymbol{0}^{\mathrm{T}} & k\beta - 1 \end{pmatrix}.$$

Since $k > 0$, the matrix $-(I_{l\times l} + k\boldsymbol{a}\boldsymbol{a}^{\mathrm{T}})$ is negative definite, so all of its eigenvalues are in the left-half plane. Also, since $k\beta = \frac{5}{4}$, the last term on the diagonal, which also indicates the last, simple eigenvalue, is in the right-half plane.

When $\theta = 1$,

$$B(1) = \begin{pmatrix} -(I_{l\times l} + k\boldsymbol{a}\boldsymbol{a}^{\mathrm{T}}) & \alpha\tilde{\boldsymbol{y}} \\ \alpha\tilde{\boldsymbol{x}}^{\mathrm{T}} & k\beta - 1 \end{pmatrix}$$

$$= -I_{(l+1)\times(l+1)} + A_0.$$

But, all eigenvalues of A_0 are strictly inside the unit circle, because of its stability. So, $-I + A_0$ has all of its eigenvalues in the left-half plane.

Now, the eigenvalues of $B(\theta)$ are symmetric with respect to the real axis, and they are continuous with respect to θ. So, when θ varies from 0 to 1, we move from a configuration of a simple eigenvalue alone on the right-half plane to all eigenvalues on the left-half plane.

This means that for some $\theta_0 \in (0,1)$, $B(\theta_0)$ has a zero eigenvalue. Lemma 3 states that this zero eigenvalue implies that

$$(\alpha\theta_0)^2 \tilde{\boldsymbol{x}}^{\mathrm{T}} \left(I - \frac{k}{k\boldsymbol{a}^{\mathrm{T}}\boldsymbol{a}+1} \boldsymbol{a}\boldsymbol{a}^{\mathrm{T}} \right) \tilde{\boldsymbol{z}} = -(k\beta - 1),$$

where $\tilde{\boldsymbol{z}} = -\tilde{\boldsymbol{y}}$.

Now, since $\theta_0 \in (0,1)$, then

$$\tilde{\boldsymbol{x}}^{\mathrm{T}} \left(I - \frac{k}{k\boldsymbol{a}^{\mathrm{T}}\boldsymbol{a}+1} \boldsymbol{a}\boldsymbol{a}^{\mathrm{T}} \right) \tilde{\boldsymbol{z}} > \frac{(k\beta - 1)}{\alpha^2}. \tag{8}$$

The matrix $(I - (k/(k\boldsymbol{a}^{\mathrm{T}}\boldsymbol{a}+1))\boldsymbol{a}\boldsymbol{a}^{\mathrm{T}})$ is symmetric and positive definite, which makes the left side of the inequality a convex function with respect to $\tilde{\boldsymbol{x}}$ and $\tilde{\boldsymbol{z}}$. Therefore, the maximum expression of the left-hand side over all possible vectors, $\boldsymbol{x}, \boldsymbol{z} \in [-1,1]^l$ is reached when $\boldsymbol{x} = \boldsymbol{z}$. The maximum of a convex function over a bounded polyhedron is attained at an extremum, therefore, the maximum is attained when the elements of $\boldsymbol{x}$ are at the boundaries of the unit interval:

$$\begin{aligned}
&\max_{x,z \in [-1,1]^l} \boldsymbol{x}^{\mathrm{T}} \left(I - \frac{k}{k\boldsymbol{a}^{\mathrm{T}}\boldsymbol{a}+1} \boldsymbol{a}\boldsymbol{a}^{\mathrm{T}} \right) \boldsymbol{z} \\
&= \max_{x \in [-1,1]^l} \boldsymbol{x}^{\mathrm{T}} \left(I - \frac{k}{k\boldsymbol{a}^{\mathrm{T}}\boldsymbol{a}+1} \boldsymbol{a}\boldsymbol{a}^{\mathrm{T}} \right) \boldsymbol{x} \\
&= \max_{x \in \{-1,1\}^l} \boldsymbol{x}^{\mathrm{T}} \left(I - \frac{k}{k\boldsymbol{a}^{\mathrm{T}}\boldsymbol{a}+1} \boldsymbol{a}\boldsymbol{a}^{\mathrm{T}} \right) \boldsymbol{x} \\
&= l - \frac{k}{k\boldsymbol{a}^{\mathrm{T}}\boldsymbol{a}+1} \min_{x \in \{-1,1\}^l} (\boldsymbol{x}^{\mathrm{T}}\boldsymbol{a})^2.
\end{aligned} \tag{9}$$

Combining (8) and (9) results in

$$l - \frac{k}{k\boldsymbol{a}^{\mathrm{T}}\boldsymbol{a}+1} \min_{x \in \{-1,1\}^l} (\boldsymbol{x}^{\mathrm{T}}\boldsymbol{a})^2 > \frac{(k\beta-1)}{\alpha^2}$$

$$\Rightarrow \left(l - \frac{k\beta - 1}{\alpha^2} \right) \frac{k\boldsymbol{a}^{\mathrm{T}}\boldsymbol{a}+1}{k} > \min_{x \in \{-1,1\}^l} (\boldsymbol{x}^{\mathrm{T}}\boldsymbol{a})^2.$$

With the constraint on α, the left side of the inequality is less than 1, but the right side is a non-negative integer. Thus

$$\min_{x \in \{-1,1\}^l} (\boldsymbol{x}^{\mathrm{T}}\boldsymbol{a})^2 = 0.$$

Assume that the maximum of (9) is obtained for $\boldsymbol{x}^{\mathrm{T}} = (x_1, \ldots, x_l)$, a set $\{t_i\}_{i=1,\ldots,l}$, that satisfies (1), is obtained by setting $t_i = x_i$ for $i = 1, \ldots, l$. □

3. Corollaries that stem from SMIUIF

Automatic power control is becoming increasingly prevalent in wireless systems. In such a system, the successful attainment of an optimal set of transmission powers depends on the spectral radius of the system gain matrix, which depends on pathgains, SINR levels, and antenna array weights. The system designer wishes to provide the highest allowable SINR level to each user. The constraint is the spectral radius of the system gain matrix, which should have a norm less than 1.

This problem leads to the question of modifying the SINR constraints for users in order to comply with this requirement, which equates to perturbing the original matrix by adding a linear combination of rank-one matrices to it [6]: Let k, l be positive integers, A_0 a real, $(l+1) \times (l+1)$ matrix with rational entries, and $\{A_i\}_{i=1,\ldots,k}$ real, $(l+1) \times (l+1)$ matrices with rational entries and rank 1. Does there exist a set of scalars, $\{t_i\}_{i=1,\ldots,k}$, from the unit interval such that the sum

$$A = A_0 + \sum_{i=1}^{k} t_i A_i \tag{10}$$

has all eigenvalues inside the unit circle? Recall we denoted this problem as Power-P. With the result of Theorem 1, we can use the same arguments as in [1] to show that problem SMIUIF can be reduced to Power-P, for the discrete-time case. Therefore, Power-P is NP-hard.

Theorem 1 can also be used to prove the discrete-time version of other NP-hard problems such as *Stable Matrix in Interval Family*. This problem consists of two sets of rational numbers $\{a^*_{i,j}\}_{1 \leqslant i,j \leqslant n}$ and $\{a^\dagger_{i,j}\}_{1 \leqslant i,j \leqslant n}$, and the question is: *Does there exist a stable matrix* $A(i,j) = a_{i,j}$ *with* $\{a^*_{i,j}\}_{1 \leqslant i,j \leqslant n} \leqslant a_{i,j} \leqslant \{a^\dagger_{i,j}\}_{1 \leqslant i,j \leqslant n}$? [1].

4. Discussion

In this paper, we have shown that the stable matrix in unit interval family problem is NP-hard for discrete-time stability. We have also discussed how this problem is closely related to some problems in wireless communications, such as automatic power control. We have also noted that SMIUIF can be used to prove NP-hardness for another problem, *Stable Matrix in Interval Family*.

References

[1] V. Blondel, J. Tsitsiklis, NP-hardness of some linear control design problems, SIAM J. Control Optim. 35 (6) (1997) 2118–2127.

[2] M. Garey, D. Johnson, Computers and Intractability: A Guide to the Theory of NP-Completeness, Freeman, New York, 1979.

[3] D. Hristu, Stabilization of LTI systems with communications constraints, Proceedings of the American Control Conference, June 2000, pp. 2342–2346.

[4] D. Hristu, K. Morgansen, Limited communication control, Systems Control Lett. 37 (4) (July 1999) 193–205.

[5] K. Mehlhorn, Graph Algorithms and NP-Completeness, Springer, Berlin, 1984.

[6] A. Mercado, K.J.R. Liu, Adaptive QoS for mobile multimedia applications using power control and smart antennas, Proceedings of the IEEE International Conference on Communications 2000, New Orleans, June 2000.

[7] A. Nemerovskii, Several NP-hard problems arising in robust stability analysis, Math. Control Signals Systems 6 (2) (1993) 99–105.

[8] F. Rashid-Farrokhi, L. Tassiulas, K.J.R. Liu, Joint optimal power control and beamforming in wireless networks using antenna arrays, IEEE Trans. Commun. 46 (10) (1998) 1313–1323.

[9] W. Rugh, Linear System Theory, Prentice-Hall, Englewood Cliffs, NJ, 1996.

[10] G. Stewart, J. Sun, Matrix Perturbation Theory, Academic Press, London, 1990.