A Simple and Unified Proof of Dyadic Shift Invariance and the Extension to Cyclic Shift Invariance

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***Abstract*— In this paper, a simple and unified proof of the dyadic shift invariance and the extension to cyclic shift invariance are presented. First, the concept of the dyadic shift invariance (DSI) and cyclic shift invariant (CSI) functions is proposed. Basic properties of the DSI and CSI functions are considered. Then, we can show that the Walsh–Hadamard transform (WHT) and discrete Fourier transform (DFT) are, in fact, special cases of the DSI and CSI functions, respectively. Many properties of the WHT and DFT can then be obtained easily from DSI and CSI points of view. The proposed unified approach is simple and rigorous. We will show that the properties of the WHT and DFT are the consequence of the basic principles of the DSI and CSI functions.**

I. INTRODUCTION

THERE are many signal processing applications where the use of effective transformation such as the Walsh–Hadamard transform (WHT) and discrete Fourier transform (DFT) is essential [1]–[5]. Basically, there are three different kinds of orderings for the WHT, specifically, the Walsh ordering, the Hadamard ordering, the Dyadic/Paley ordering [1], [2], and the Cal–Sal ordering [7]. It is well known that the power spectrum of the WHT is invariant to a dyadic shift of the data sequence. Unfortunately, up to now, there is still no unified proof of the invariance for all of the orderings. The only proof available is of show-by-example type illustration [1], [2], [4], [5] which, though demonstrates the invariant property, does not provide a rigorous treatment of the dyadic shift invariance. As an example, let $\{x\}$ be an N-periodic sequence and $\{x_k\}$ be the sequence subjects to a dyadic shift of size k, and $\{X\}$ be the WHT of $\{x\}$ and $\{X_k\}$ be the WHT of $\{x_k\}$, respectively. When $N = 8$ and $k = 1$, it can be shown [1, p. 117] that

$$\{X_1\} = \text{diag}\,\{1, -1, 1, -1, 1, -1, 1, -1\} \cdot \{X\}$$

where diag $\{1, -1, 1, -1, 1, -1, 1, -1\}$ is a diagonal matrix with diagonal elements $1, -1, 1, -1, 1, -1, 1, -1$. Therefore the power spectrum remains the same after the dyadic shift. This property is well known and well accepted. However, a simple proof for general N and k is still not available. In the classroom teaching, students always have questions such as: how do we know if $N = 128$ will work? Do we have to show that all different k will work?

Manuscript received June 1991; revised February 1992.
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IEEE Log Number 9211753.

To answer these questions, here, we not only propose a simple but rigorous treatment of the subject, but also provide a more general way to look into various properties of the WHT. We first propose the concept of dyadic decomposition and present some properties of the dyadic shift invariant function. Then, based on these, we will show that all the WHT's are dyadic shift invariant functions and present unified treatments for various known properties of the WHT. This unified approach provides a deep insight into various properties of the WHT. The results can be easily extended to other transform functions such as DFT which, on the other hand, is a cyclic shift invariant function. This extension is also considered in this paper. We will see that all the well-known properties of the WHT and DFT are the consequence of the basic principles of the special functions. In fact, the WHT is a special case of the dyadic shift invariant function and the DFT is a special case of a cyclic shift invariant function.

The dyadic shift invariance is presented in Section II followed by the cyclic shift invariance in Section III.

II. DYADIC SHIFT INVARIANCE

A function $h(m,n)$ is said to be *dyadic decomposable* if it satisfies

$$h(m \oplus k, n) = h(m,n) \cdot h(k,n) \tag{1}$$

where $\oplus$ is the modulo 2 addition. A transformation is said to be *dyadic shift invariant* (DSI) if the transform function is dyadic decomposible with unity norm. That is

1. $||h(m,n)||^2 = h(m,n) \cdot h^*(m,n) = 1$
2. $h(m \oplus k, n) = h(m,n) \cdot h(k,n)$

where * is the complex conjugate operation. Let $\{x(m)\}$ be a real-valued N-periodic sequence and $\{X(n)\}$ be the DSI transformation of $\{x(m)\}$. We have

$$X(n) = \frac{1}{N} \sum_{m=0}^{N-1} x(m) h(m,n). \tag{2}$$

Theorem 1: The power spectrum of a DSI transform function is dyadic shift invariant.

Proof: Let $\{X_k(n)\}$ be the transformation of $\{x(m \oplus k)\}$, where $\{x(m \oplus k)\}$ is the sequence obtained by subjecting $\{x(m)\}$ to a dyadic shift of size k. Since the modulo 2 addition

TABLE I
TTRUTH TRUTH FOR $(m_s \oplus k_s)n_s$ AND $m_s n_s \oplus k_s n_s$

m_s	l_s	n_s	$m_s \oplus l_s$	$m_s n_s$	$l_s n_s$	$(m_s \oplus l_s)n_s$	$m_s n_s \oplus l_s n_s$
0	0	0	0	0	0	0	0
0	0	1	0	0	0	0	0
0	1	0	1	0	0	0	0
0	1	1	1	0	1	1	1
1	0	0	1	0	0	0	0
1	0	1	1	1	0	1	1
1	1	0	0	0	0	0	0
1	1	1	0	1	1	0	0

TABLE II
TRUTH TABLE FOR $(-1)^{m_s \oplus n_s}$ AND $(-1)^{m_s+n_s}$

m_s	n_s	$m_s \oplus n_s$	$m_s + n_s$	$(-1)^{m_s \oplus n_s}$	$(-1)^{m_s+n_s}$
0	0	0	0	1	1
0	1	1	1	-1	-1
1	0	1	1	-1	-1
1	1	0	2	1	1

is the same operation as the modulo 2 subtraction, one obtains

$$X_k(n) = \frac{1}{N}\sum_{m=0}^{N-1} x(m \oplus k)h(m,n)$$
$$= \frac{1}{N}\sum_{\hat{m}=0}^{N-1} x(\hat{m})h(\hat{m} \oplus k, n).$$

Since $h(m,n)$ is DSI, we have

$$X_k(n) = \frac{1}{N}\sum_{\hat{m}=0}^{N-1} x(\hat{m})h(\hat{m},n)h(k,n) = X(n)h(k,n)$$

and the power spectrum of $\{X_k(n)\}$ is

$$||X_k(n)||^2 = ||X(n)||^2 \cdot ||h(k,n)||^2 = ||X(n)||^2.$$

The power spectrum is dyadic shift invariant. □

Let m, n be any two real-valued parameters which have the binary representation

$$m = m_{N-1}2^{N-1} + m_{N-2}2^{N-2} + \cdots + m_1 2^1 + m_0 2^0$$
$$n = n_{N-1}2^{N-1} + n_{N-2}2^{N-2} + \cdots + n_2 2^1 + n_0 2^0. \quad (3)$$

The bit-valued inner product $\langle m, n\rangle$ is defined as

$$\langle m, n\rangle = \sum_{s=0}^{N-1} m_s n_s. \quad (4)$$

Lemma 1: The WHT function $(-1)^{\langle m,n\rangle}$ is DSI.

Proof: Since $(-1)^{\langle m \oplus k, n\rangle} = (-1)^{\Sigma_{s=0}^{N-1}(m_s \oplus k_s)n_s}$, and from Table I, the Boolean function $(m_s \oplus k_s)n_s$ is equivalent to the function $m_s n_s \oplus k_s n_s$. We have

$$(-1)^{\langle m \oplus k, n\rangle} = (-1)^{\Sigma_{s=0}^{N-1} m_s n_s \oplus k_s n_s} = (-1)^{\langle m,n\rangle \oplus \langle k,n\rangle}. \quad (5)$$

From Table II, it can be found

$$(-1)^{\langle m,n\rangle \oplus \langle k,n\rangle} = (-1)^{m_1 n_1 \oplus k_1 n_1} \cdot (-1)^{m_2 n_2 \oplus k_2 n_2}$$
$$\cdots (-1)^{n_{N-1} n_{N-1} \oplus k_{N-1} n_{N-1}}$$
$$= (-1)^{m_1 n_1 + k_1 n_1} \cdot (-1)^{m_2 n_2 + k_2 n_2}$$
$$\cdots (-1)^{n_{N-1} n_{N-1} + k_{N-1} n_{N-1}}$$
$$= (-1)^{\langle m,n\rangle} \cdot (-1)^{\langle k,n\rangle}. \quad (6)$$

As $||(-1)^{\langle m,n\rangle}||^2 = 1$, one concludes that WHT function is a DSI function. □

In general, the WHT is defined as

$$X(n) = \frac{1}{N}\sum_{m=0}^{N-1} x(m)(-1)^{\langle m, r(n)\rangle} \quad (7)$$

where $r(n)$ is a function which depends on the ordering of WHT. Let

$$\langle m, r(n)\rangle = \sum_{s=0}^{N-1} m_s r_s(n)$$

where $r_s(n), s = 0, 1, \cdots, N-1$ is the binary representation of $r(n)$ as in (3). For the Hadamard ordering, $r_s(n) = n_s$. For the Walsh ordering,

$$r_s(n) = \begin{cases} n_{N-s} + n_{N-s-1} & \text{for } s \neq 0 \\ n_{N-1} & \text{for } s = 0. \end{cases}$$

For the Dyadic/Paley ordering, $r_s(n) = n_{N-1-s}$. We have the following theorem.

Theorem 2: The power spectrum of all orderings of the WHT are dyadic shift invariant.

Proof: By Lemma 1, we know the function $(-1)^{\langle m, r(n)\rangle}$ is DSI. From Theorem 1, the power spectrum of WHT is dyadic shift invariant. □

Lemma 2: Let $\{X\}$ be the WHT of $\{x\}$ and $\{X_k\}$ be the transform of $\{x\}$ subjected to a dyadic shift of size k, where $\{x\}$ is an N-periodic sequence. Then the relationship between X_k and X is

$$X_k(n) = X(n) \cdot (-1)^{\langle k,n\rangle}. \quad (8)$$

Proof: Since $X_k(n) = 1/N\,\Sigma_{m=0}^{N-1} x(m \oplus k)(-1)^{\langle m,n\rangle}$, by Lemma 1, we have

$$X_k(n) = \frac{1}{N}\sum_{\hat{m}=0}^{N-1} x(\hat{m})(-1)^{\langle \hat{m},n\rangle}(-1)^{\langle k,n\rangle}$$
$$= X(n) \cdot (-1)^{\langle k,n\rangle} \square$$

With Lemma 2, we can interpret that for the WHT, the dyadic shift in the time domain results in a "phase shift" of either 0 or π in the frequency domain.

The dyadic cross-correlation function of two real-valued N-periodic sequences $\{x(m)\}$ and $\{y(m)\}$ is defined as

$$z(m) = \frac{1}{N}\sum_{h=0}^{N-1} x(h)y(m \oplus h). \quad (9)$$

Let the DSI transform of $z(m)$ be $Z(n)$ and the DSI transform of sequences $x(m)$ and $y(m)$ be $X(n)$ and $Y(n)$, respectively. We have

$$Z(n) = DSIT(z(m)) = \frac{1}{N^2}\sum_{m=0}^{N-1}\sum_{l=0}^{N-1} x(l)y(m \oplus l)h(m,n)$$
$$= \frac{1}{N^2}\sum_{m=0}^{N-1}\sum_{l=0}^{N-1} x(l)y(m \oplus l)h(m,n)h(l,n)h^*(l,n) \quad (10)$$
$$Z(n) = \frac{1}{N^2}\sum_{l=0}^{N-1} x(l)h^*(l,n)\sum_{m=0}^{N-1} y(m \oplus l)h(m \oplus l, n)$$
$$= \overline{X}^*(n) \cdot Y(n) \quad (11)$$

where $\overline{X}^* \triangleq (DSI(x^*))^*$. If it is a real DSI function such as WHT, then $Z(n) = X(n) \cdot Y(n)$. Obvious, the power spectrum is dyadic shift invariant in the way that

$$||Z(n)||^2 = ||X(n)||^2 \cdot ||Y(n)||^2. \tag{12}$$

A 2–D transformation which is said to be DSI if the transform function satisfies

1. $||h(m_1, m_2, n_1, n_2)||^2 = 1$
2. $h(m_1 \oplus k, m_2 \oplus l, n_1, n_2) = h(m_1, m_2, n_1, n_2) \cdot h(k, l, n_1, n_2)$.

Analogous to the 1-D case, it can be shown that the power spectrum of a two-dimensional (2-D) DSI transform function is dyadic shift invariant. The 2-D WHT function is $(-1)^{\langle m_1, n_1\rangle + \langle m_2, n_2\rangle}$. Since

$$\begin{aligned}(-1)^{\langle m_1 \oplus k, n_1\rangle + \langle m_2 \oplus l, n_2\rangle} &= (-1)^{\langle m_1, n_1\rangle}(-1)^{\langle k, n_1\rangle}(-1)^{\langle m_2, n_2\rangle}(-1)^{\langle l, n_2\rangle}\\ &= (-1)^{\langle m_1, n_1\rangle + \langle m_2, n_2\rangle} \cdot (-1)^{\langle k, n_1\rangle + \langle l, n_2\rangle}\end{aligned} \tag{13}$$

the 2-D WHT function is a 2-D DSI function. Therefore, all the properties of the 2-D WHT follow as discussed in the 1-D case. For instance, denote $X_{k,l}(n_1, n_2)$ as the transformation of $\{x(m_1 \oplus k, m_2 \oplus l)\}$ which is the sequence obtained by subjecting $\{x(m_1, m_2)\}$ to a dyadic shift of size k in m_1 direction and of size l in m_2 direction. It can be easily obtained that

$$X_{k,l}(n_1, n_2) = X(n_1, n_2)(-1)^{\langle n_1, k\rangle + \langle n_2, l\rangle}. \tag{14}$$

From the analogy in the 1-D case, the 2-D cross-correlation function can be defined as

$$\begin{aligned}z(m_1, m_2) = \frac{1}{N_1 N_2} \sum_{h_2=0}^{N_2-1} \sum_{h_1=0}^{N_1-1} x(m_1 \oplus h_1, m_2 \oplus h_2)\\ \cdot y(h_1, h_2).\end{aligned} \tag{15}$$

Let the 2-D DSI transform of z, x, and y be Z, X, and Y, respectively. With the same derivation as before, we can show that

$$Z(n_1, n_2) = \overline{X}^*(n_1, n_2) \cdot Y(n_1, n_2). \tag{16}$$

Again, if the 2-D DSI transform function is real-valued such as 2-D WHT, then $Z(n_1, n_2) = X(n_1, n_2) \cdot Y(n_1, n_2)$.

III. Cyclic Shift Invariance

The above results can be extended to cyclic shift invariance. A N-periodic function $g(m, n)$ is *cyclic decomposable* if it satisfies

$$g(m + k, n) = g(m, n) \cdot g(k, n). \tag{17}$$

A transformation is said to be cyclc shift invariant (CSI) if the transform function is cyclic decomposable with unity norm. That is

1) $||g(m, n)||^2 = 1$
2) $g(m + k, n) = g(m, n) \cdot g(k, n)$.

Theorem 3: The power spectrum of a CSI transform function is cyclic shift invariant.

Proof: Let $\{x(m)\}$ be a real-valued N-periodic sequence and $\{X(n)\}$ be transformation of $\{x(m)\}$. We have

$$X(n) = \frac{1}{N} \sum_{m=0}^{N-1} x(m) h(m, n). \tag{18}$$

Let $\{X_k(n)\}$ be the transformation of $\{x(m + k)\}$, where $\{x(m + k)\}$ is the sequence obtained by subjecting $\{x(m)\}$ to a cyclic shift of size k. One obtain

$$\begin{aligned}X_k(n) &= \frac{1}{N} \sum_{m=0}^{N-1} x(m + k) h(m, n)\\ &= \frac{1}{N} \sum_{\hat{m}=0}^{N-1} x(\hat{m}) h(\hat{m} - k, n).\end{aligned} \tag{19}$$

Since $h(m, n)$ is cyclic decomposible, we have

$$X_k(n) = \frac{1}{N} \sum_{\hat{m}=0}^{N-1} x(\hat{m}) h(\hat{m}, n) h(-k, n) = X(n) h(-k, n).$$

By the property that the norm of $h(m, n)$ equals unity,

$$||X_k(n)||^2 = ||X(n)||^2 \cdot ||h(-k, n)||^2 = ||X(n)||^2. \square$$

The cyclic cross-correlation (or convolution) function of two real-valued N-periodic sequences $\{x(m)\}$ and $\{y(m)\}$ is defined as

$$z(m) = \frac{1}{N} \sum_{h=0}^{N-1} x(h) y(m + h). \tag{20}$$

Let the CSI transform of $z(m)$ be $Z(n)$ and the CSI transform of sequences $x(m)$ and $y(m)$ be $X(n)$ and $Y(n)$, respectively. We have

$$\begin{aligned}Z(n) = CSIT(z(m)) &= \frac{1}{N^2} \sum_{m=0}^{N-1} \sum_{l=0}^{N-1} x(l) y(m + l) h(m, n)\\ &= \frac{1}{N^2} \sum_{m=0}^{N-1} \sum_{l=0}^{N-1} x(l) y(m + l) h(m, n) h(l, n) h^*(l, n)\end{aligned} \tag{21}$$

$$\begin{aligned}Z(n) &= \frac{1}{N^2} \sum_{l=0}^{N-1} x(l) h^*(l, n) \sum_{m=0}^{N-1} y(m + l) h(m + l, n)\\ &= \overline{X}^*(n) \cdot Y(n)\end{aligned} \tag{22}$$

where $\overline{X}^* \triangleq (\mathrm{CSI}(x^*))^*$. It can be easily shown that the discrete Fourier transform (DFT) function $\exp(-j2\pi kn/N)$ is a CSI function. Therefore, the power spectrum of DFT is cyclic shift invariant. Many well-known properties of the DFT can then be easily obtained by the same derivations as in Section II.

IV. Conclusion

Simple and unified proofs of basic properties of the dyadic shift invariance and the cyclic shift invariance are presented in this paper. As we have shown, the WHT and DFT are the special cases of the DSI and CSI functions, respectively. Therefore, all the properties of the DSI and CSI functions are preserved in the WHT and DFT, respectively. Many properties are then easily derived by using this approach. In conclusion, the properties of the WHT and DFT are the consequence of the basic principles of the DSI and CSI functions.

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Analysis of Robot Dynamics and Compensation Using Classical and Computed Torque Techniques

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***Abstract*—A classical analysis of the dynamics of robot manipulators is presented. It is shown that these systems have configuration-dependent properties and can be open-loop unstable. Due to this fact, present day linear controllers are inefficient. On the other hand, nonlinear hardware and software compensation methods also are shown to have some limitations. Controllers based on the direct design algorithm and the computed torque method have superior performances. These algorithms have nonlinear loops yet, our paper shows that a linear analysis is still feasible. Therefore, classical design tools can be adopted in order to develop practical implementations.**

VI. INTRODUCTION

THE dynamics of robot manipulators is highly nonlinear which makes difficult their efficient control. Classical control methods are well known; however, they are inadequate in the presence of strong nonlinearities. On the other hand, nonlinear controllers [1]–[4] produce better results but the nonlinear analysis and design is not as systematic and clear as the linear case. Some work has been done on relating linear methods to manipulator dynamics [5]–[14]. However, the complexity of the problem has not allowed yet methods which permit general conclusions to be drawn about stability, imperfect modeling effects, etc. This paper intends to link classical linear methods with robot modern nonlinear control schemes. Having this idea in mind we organize the paper as follows. In Section II we analyze the dynamics of a two degrees of freedom (d.o.f.) manipulator from a classical (Laplace-based) point of view. Using this approach we derive a set of transfer functions (TF's) that characterize the dynamics of robot manipulators. The TF's reveal that manipulating systems are intrinsically unstable. Therefore, in order to render the system stable, we need appropriate compensation techniques. In this line of thought, in Section III, we analyze both hardware and software compensation methods. These compensations have limitations which make necessary the development of complementary control strategies. In Section IV we analyze, from a classical perspective, model-based nonlinear algorithms that accomplish not only a dynamic compensation but also the control action. Finally, in Section V, conclusions are drawn.

VII. DYNAMICS OF A TWO DEGREES OF FREEDOM MANIPULATOR

The dynamic equations of the two d.o.f. manipulator (Fig. 1) can be easily obtained from the Lagrangian [15], [16]

Manuscript received June 1991; revised April 1992.
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IEEE Log Number 9211758.