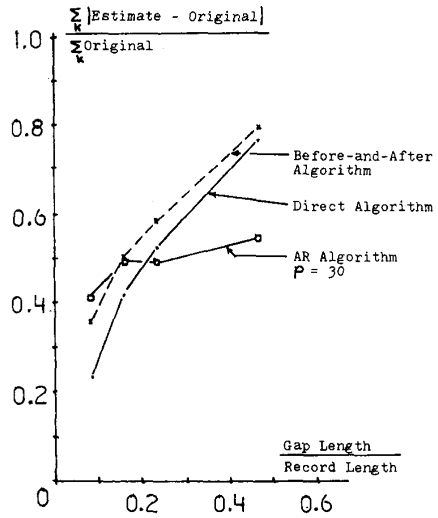

Fig. 4. Errors in spectral estimates for damped-sine-wave data set.

pendent of gap length over the range tested and were the best for gap lengths greater than 20%.

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Fast Orthogonalization Algorithm and Parallel Architecture for AR Spectral Estimation Based on Forward-Backward Linear Prediction

K. J. Ray Liu and S. F. Hsieh

***Abstract*—The truncated *QR* methods have been shown to be comparable to the SVD-based methods for the sinusoidal frequency estimation based on the forward-backward linear prediction (FBLP) model. However, without exploiting the special structure of the FBLP matrix, the *QR* decomposition (QRD) of the FBLP matrix has the computational complexity on the order of $2(6m - n)n^2/3 + O(n^2)$ for a $2m \times n$ FBLP matrix. Here we propose a fast algorithm to perform the QRD of the FBLP matrix by exploiting its special Toeplitz-Hankel form. The computational complexity is then reduced to $10n^2 + 4mn + O(n)$. The fast algorithm can also be easily implemented onto a linear systolic array. The number of time steps required is further reduced to $2m + 5n - 4$ by using the parallel implementation.**

I. Introduction

High-resolution spectral estimation is an important subject in many applications of modern signal processing. The fundamental problem in applying various high-resolution spectral estimation algorithms is the computational complexity. In the pioneering paper of Tufts and Kumaresan [1], a SVD-based method for solving the forward-backward linear prediction (FBLP) least squares (LS) problem was used to resolve the frequencies of closely spaced sinusoids from a limited amount of data samples. By imposing an excessive order in the FBLP model and then truncating small singular values to zero, this truncated SVD method yields a low signal-to-noise ratio (SNR) threshold and greatly suppresses spurious frequencies. However, the massive computation required by SVD makes it unsuitable for *real-time* superresolution applications.

Recently, the truncated *QR* methods [4] have been shown to be comparable to the SVD-based methods in various situations. It is very effective for the sinusoidal frequency estimation based on the FBLP model. However, without considering the special structure of the FBLP matrix, the *QR* decomposition (QRD) of the FBLP matrix still has the computational complexity on the order of $O(n^3)$.

Manuscript received April 21, 1991; revised May 12, 1992.

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IEEE Log Number 9206011.

Seeking fast algorithms for specially structured matrices has captured lots of attention recently, especially the Toeplitz-structured matrices are used in many signal processing applications [2], [3], [7]–[10]. However, exploiting the special structure of the FBLP matrix for fast algorithm implementation has not yet been considered so far. Here we propose a fast algorithm to perform the QRD of the FBLP matrix. The computational cost of the truncated QR methods can be further reduced from $O(n^3)$ to $O(n^2)$ which makes it more attractive than the SVD-based methods. Without exploiting the special structure of the FBLP matrix, the straightforward QRD of the FBLP matrix has the computational complexity on the order of $2(6m - n)n^2/3 + O(n^2)$ for a $2m \times n$ FBLP matrix. The proposed fast algorithm reduces it to the order of $10n^2 + 4mn + O(n)$. We will also show that the proposed fast algorithm is amendable to parallel processing. A fully pipelined linear systolic array based on the multiphase operations is used to implement the fast algorithm parallelly. The required time steps is further reduced to $2m + 5n - 4$.

The idea of FBLP was originated by Burg [5] for the lattice predictors. To improve the performance, Tufts and Kumaresan [1] developed a modified FBLP method which is very effective for estimating closely spaced frequencies. The FBLP method is to minimize the sum of the FBLP errors energy,

$$\min_{w} \sum_{i=M+1}^{N} [|e_f(i)|^2 + |e_b(i)|^2] \tag{1}$$

where e_f and e_b are the forward, and backward residuals, respectively. The FBLP method is to solve the LS problem [13]:

$$\min_{w} \|Aw - b\|^2 \tag{2}$$

with

$$K = [A \vdots b] = \begin{bmatrix} H \\ \hdashline T \end{bmatrix} = \begin{bmatrix} TJ \\ \hdashline T \end{bmatrix} = \begin{bmatrix} k_1 \\ k_2 \\ \vdots \\ k_{2(N-M)} \end{bmatrix} \tag{3}$$

and

$$T = \begin{bmatrix} u(M+1) & u(M) & \cdots & u(2) & u(1) \\ \vdots & \vdots & & \vdots & \vdots \\ u(N-1) & u(N-2) & \cdots & u(N-M) & u(N-M-1) \\ u(N) & u(N-1) & \cdots & u(N-M+1) & u(N-M) \end{bmatrix} \tag{4}$$

where J is an exchange matrix. The matrix of the form as given in (3) is called the Toeplitz–Hankel matrix. In fact, the augmented matrix of the FBLP problem is of the Toeplitz–Hankel form with a special property, i.e., $H = TJ$. This special property can be used for developing a fast algorithm that will be considered in the following sections.

II. Exploiting the Toeplitz–Hankel Structure

In using the truncated QR method for the high-resolution AR spectral estimation, the key computational issue is to solve the FBLP LS problem based on the QR decomposition (QRD). Without considering the special structure, a conventional QRD requires $\approx 4(N - M)M^2 + O(M^2)$ multiplications to obtain the upper triangular matrix R. This is on the order of $O(NM^2)$ since usually $N \gg M$. Thus, a reasonable approach is to find a fast algorithm for the FBLP LS problem by exploiting its special Toeplitz–Hankel structure. This problem has not been considered so far, though the LS problem with Toeplitz structure has been studied extensively [2], [3], [6]–[10].

The Toeplitz part of the Toeplitz–Hankel matrix can be partitioned as

$$T = \begin{bmatrix} u(M+1) & x^T \\ y & \tilde{T} \end{bmatrix} = \begin{bmatrix} \tilde{T} & u \\ v^T & u(N-M) \end{bmatrix} \tag{5}$$

where

$$\tilde{T} = \begin{bmatrix} u(M+1) & u(M) & \cdots & u(2) \\ u(M+2) & u(M+1) & \cdots & u(3) \\ \vdots & \vdots & & \vdots \\ u(N-1) & u(N-2) & \cdots & u(N-M) \end{bmatrix}$$

$$x^T = [u(M), \cdots, u(2), u(1)]$$

$$y^T = [u(M+2), \cdots, u(N-1), u(N)]$$

$$u^T = [u(1), u(2), \cdots, u(N-M-1)]$$

$$v^T = [u(N), u(N-1), \cdots, u(N-M+1)]$$

and the Hankel part of the Toeplitz–Hankel matrix can be partitioned as

$$H = TJ = \begin{bmatrix} u & \tilde{H} \\ u(N-M) & v^{BT} \end{bmatrix} = \begin{bmatrix} x^{BT} & u(M+1) \\ \tilde{H} & y \end{bmatrix} \tag{6}$$

where

$$\tilde{H} = \tilde{T}J$$

$$v^{BT} = [u(N-M+1), \cdots, u(N-1), u(N)] = v^T J$$

and

$$x^{BT} = [u(1), u(2), \cdots, u(M)] = x^T J.$$

Here B denotes the backward arrangement of a vector.

Now, from the above partitions, the Toeplitz–Hankel matrix K can be partitioned as follows:

$$K = \begin{bmatrix} u & \tilde{H} \\ u(N-M) & v^{BT} \\ \hdashline u(M+1) & x^T \\ y & \tilde{T} \end{bmatrix} \tag{7}$$

and

$$K^T K = \begin{bmatrix} u^T u + u^2(N-M) + u^2(M+1) + y^T y & u^T \tilde{H} + u(N-M) v^{BT} + u(M+1) x^T + y^T \tilde{T} \\ \tilde{H}^T u + v^B u(N-M) + x u(M+1) + \tilde{T}^T y & \tilde{H}^T \tilde{H} + \tilde{T}^T \tilde{T} + v^B v^{BT} + x x^T \end{bmatrix}. \tag{8}$$

Also, the matrix K can be partitioned as

$$K = \begin{bmatrix} x^{BT} & u(M+1) \\ \tilde{H} & y \\ \hdashline \tilde{T} & u \\ v^T & u(N-M) \end{bmatrix} \tag{9}$$

and with this partition, we have

$$K^TK = \begin{bmatrix} x^Tx^{BT} + \tilde{H}^T\tilde{H} + \tilde{T}^T\tilde{T} + vv^T & x^Bu(M+1) + \tilde{H}^Ty + \tilde{T}^Tu + vu(N-M) \\ u(M+1)x^{BT} + y^T\tilde{H} + u^T\tilde{T} + u(N-M)v^T & u^2(M+1) + y^Ty + u^Tu + u^2(N-M) \end{bmatrix}. \tag{10}$$

Let the QRD of the matrix K be $K = QR$, where $R \in \Re^{(M+1)\times(M+1)}$ is an upper triangular matrix and it can also be partitioned as follows:

$$R = \begin{bmatrix} r_{1,1} & r_1^T \\ \mathbf{0} & R_b \end{bmatrix} = \begin{bmatrix} R_t & r_2 \\ \mathbf{0}^T & r_{M+1,M+1} \end{bmatrix} \tag{11}$$

where $R_b \in \Re^{M\times M}$ is the principal bottom submatrix of R, $R_t \in \Re^{M\times M}$ is the principal top submatrix of R, and

$$r_1^T = [r_{1,2}, r_{1,3}, \cdots, r_{1,M+1}]$$

$$r_2^T = [r_{1,M+1}, r_{2,M+1}, \cdots, r_{M,M+1}].$$

Note that both R_b and R_t are upper triangular matrices. Since the matrix Q is orthogonal, we have

$$K^TK = R^TR \tag{12}$$

and

$$R^TR = \begin{bmatrix} r_{1,1}^2 & r_{1,1}r_1^T \\ r_1r_{1,1} & r_1r_1^T + R_b^TR_b \end{bmatrix} = \begin{bmatrix} R_t^TR_t & R_t^Tr_2 \\ r_2^TR_t & r_2^Tr_2 + r_{M+1,M+1}^2 \end{bmatrix}. \tag{13}$$

Define

$$\tilde{K} = \begin{bmatrix} \tilde{H} \\ \hdashline \tilde{T} \end{bmatrix} \tag{14}$$

then we have

$$\tilde{K}^T\tilde{K} = \tilde{H}^T\tilde{H} + \tilde{T}^T\tilde{T}. \tag{15}$$

From the lower right submatrices of (8) and (13), we obtain

$$R_b^TR_b + r_1r_1^T = \tilde{K}^T\tilde{K} + xx^T + v^Bv^{BT}. \tag{16}$$

Also, from the upper left submatrices of (10) and (13), we have

$$R_t^TR_t = \tilde{K}^T\tilde{K} + vv^T + x^Bx^{BT}. \tag{17}$$

Substituting (17) to (16), we obtain the relation between R_b and R_t as given by

$$R_b^TR_b = R_t^TR_t + xx^T - x^Bx^{BT} + v^Bv^{BT} - vv^T - r_1r_1^T. \tag{18}$$

III. The Fast Algorithm

It is clear how to perform the updating and downdating of the Cholesky factors [2]. As we can see, in [18], there are two rank-1 updatings and three rank-1 downdatings. Let us split (18) into a sequence of five up/downdating equations given by

$$R_1^TR_1 = R_t^TR_t + xx^T$$

$$R_2^TR_2 = R_1^TR_1 - x^Bx^{BT}$$

$$R_3^TR_3 = R_2^TR_2 + v^Bv^{BT}$$

$$R_4^TR_4 = R_3^TR_3 - vv^T$$

$$R_b^TR_b = R_4^TR_4 - r_1r_1^T \tag{19}$$

where R_t, R_b, and R_i, $i = 1, 2, 3, 4$ are all $M \times M$ upper triangular matrices, and all of the vectors involved are M-dimensional. In order to start the above recursions, the first row of R_t (or R) must be available. In general, there is no shortcut for obtaining this row and it can be done by a sequence of Givens rotations on the matrix K to zero out the first column of K, except its leading element on the diagonal. By denoting # as a "don't care" element or vector, the fast algorithm is summarized in Table I.

As we can see, for the initialization (obtaining the first row of R), the computational cost is $\approx 4(N-M)M$ multiplications (since only half of the rotation needed to be done). Following this, the recursions in the main iterations are then started. As there are five rotation-like up/downdatings, the computational cost is

$$5 \times (4M + 4(M-1) + \cdots + 4 \cdot 1)$$

$$= 20\left(\frac{M(M+1)}{2} - 1\right) \approx 10M^2 + O(M)$$

(for multiplication). Therefore, the total computational complexity is $\approx 10M^2 + 4(N-M)M$ (for multiplication) for a $2(N-M) \times M$ Toeplitz–Hankel matrix. As mentioned before, without considering the special structure, by using the conventional QRD, the computational complexity is of $\approx 4NM^2 + O(M^2)$. Obviously, the proposed fast algorithm has an improvement of an order of magnitude. In general, for the QRD of a $2m \times n$ Toeplitz–Hankel matrix, the fast algorithm needs $10n^2 + 4mn + O(n)$ multiplications, while a conventional implementation needs $2(6m - n)n^2/3 + O(n^2)$, where $m >> n$.

It the least squares weight vector is of interested, a backward substitution can then be used for computing the weight vector. For the truncated QR method, a truncation of the noise subspace is necessary before computing the weight vector [4].

IV. Parallel Implementation

The fast algorithm obtained in the previous section not only reduces the computational complexity, but is also amenable for parallel implementation. From the fact that only the first row of the upper triangular matrix R has to be obtained first, a linear array of $M + 1$ processing cells, as shown in Fig. 1, can be used to rotate the matrix K such that the first column can be zeroed out and when the initialization phase is finished, the first row of the matrix R is kept in the linear array. Fig. 2 shows the initialization to obtain the first row of R. The operations of the processing cells are given in Table II. The data matrix is arranged in a skewed manner for the systolic array implementation. The idea is similar to the triangular array for the QRD proposed by Gentleman and Kung [11]. The difference is that their scheme is a general one without considering any special structure of the data matrix. Accordingly, a full triangular array is needed.

TABLE I
SUMMARY OF THE FAST ALGORITHM

(Initialization)

$\boldsymbol{x}^{T^{(0)}} = [u(M), u(M-1), \cdots, u(1)]$

$\boldsymbol{x}^{BT^{(0)}} = [u(1), u(2), \cdots, u(M)]$

$\boldsymbol{v}^{T^{(0)}} = [u(N), u(N-1), \cdots, u(N-M+1)]$

$\boldsymbol{v}^{BT^{(0)}} = [u(N-M+1), u(N-M+2), \cdots, u(N)]$

$\boldsymbol{k}_{(1)} = \boldsymbol{k}_1$

For $i = 1$ to $2(N-M)-1$,

$$\begin{bmatrix} \boldsymbol{k}_{(i+1)} \\ \# \end{bmatrix} = U_{(i)} \begin{bmatrix} \boldsymbol{k}_{(i)} \\ \boldsymbol{k}_{i+1} \end{bmatrix}$$

End For;

$[r_{1,1}, \boldsymbol{r}_1^T] = [\boldsymbol{r}_1^t, \#]^T = \boldsymbol{k}_{(2(N-M))}^T$

$\boldsymbol{r}_1^{T^{(0)}} = \boldsymbol{r}_1^{\mathrm{T}}.$

(Main Iterations)

For $i = 1$ to $M-1$,

(Phase 1)

$$\begin{bmatrix} \boldsymbol{r}_i^{1^T} \\ 0, \boldsymbol{x}^{T(i)} \end{bmatrix} = U_{(i)}^1 \begin{bmatrix} \boldsymbol{x}^{T(i-1)} \\ \boldsymbol{r}_i^{t^T} \end{bmatrix}$$

(Phase 2)

$$\begin{bmatrix} \boldsymbol{r}_i^{2^T} \\ 0, \boldsymbol{x}^{BT(i)} \end{bmatrix} = \tilde{U}_{(i)}^2 \begin{bmatrix} \boldsymbol{x}^{BT(i-1)} \\ \boldsymbol{r}_i^{1^T} \end{bmatrix}$$

(Phase 3)

$$\begin{bmatrix} \boldsymbol{r}_i^{3^T} \\ 0, \boldsymbol{v}^{BT(i)} \end{bmatrix} = U_{(i)}^3 \begin{bmatrix} \boldsymbol{v}^{BT(i-1)} \\ \boldsymbol{r}_i^{2^T} \end{bmatrix}$$

(Phase 4)

$$\begin{bmatrix} \boldsymbol{r}_i^{4^T} \\ 0, \boldsymbol{v}^{T(i)} \end{bmatrix} = \tilde{U}_{(i)}^4 \begin{bmatrix} \boldsymbol{v}^{T(i-1)} \\ \boldsymbol{r}_i^{3^T} \end{bmatrix}$$

(Phase 5)

$$\begin{bmatrix} \boldsymbol{r}_i^{b^T} \\ 0, \boldsymbol{r}_1^{T(i)} \end{bmatrix} = \tilde{U}_{(i)}^5 \begin{bmatrix} \boldsymbol{r}^{T(i-1)} \\ \boldsymbol{r}_i^{4^T} \end{bmatrix},$$

$\boldsymbol{v}_{i+1}^{t^T} = \boldsymbol{r}_i^{b^T}$ excluding the last one,

End For.

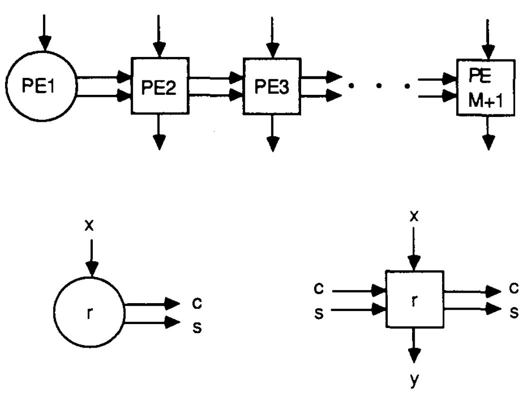

Fig. 1. The linear systolic array and its processing cells.

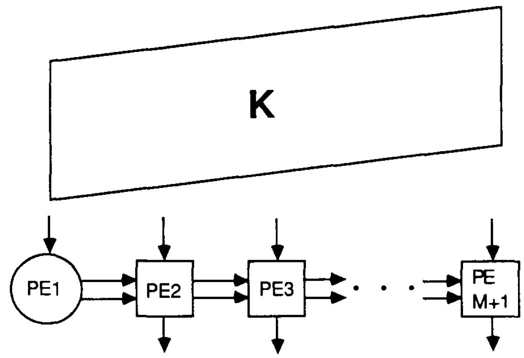

Fig. 2. The initialization.

Due to the consideration of the special Toeplitz–Hankel structure, once the first row of the matrix R is available, the subsequent rows of R can be generated one by one by the main iterations given in the fast algorithm. To start the main iterations, $\boldsymbol{r}_1^{t^T}$ is needed. Fortunately, it is the first M elements of the first row of R that are stored in the first M processing cells. The main iterations are now started with inputs $\boldsymbol{x}^{T^{(0)}}$, $\boldsymbol{x}^{BT^{(0)}}$, $\boldsymbol{x}^{BT^{(0)}}$, $\boldsymbol{v}^{T^{(0)}}$, and $\boldsymbol{r}_1^{T^{(0)}}$, and the outputs are $\boldsymbol{x}^{T^{(1)}}$, $\boldsymbol{x}^{BT^{(1)}}$, $\boldsymbol{x}^{BT^{(1)}}$, $\boldsymbol{v}^{T^{(1)}}$, and $\boldsymbol{r}_1^{T(1)}$, respectively, as illustrated in Fig. 3. The outputs have one less dimension than their inputs do. The vector $\boldsymbol{r}_1^{b^T}$ can now be obtained on the linear array.

As given in the main iterations, there are five different phases. The operations of the processing cells for different phases are given in Table II. Based on the multiphase concept proposed in [12], the outputs are fedback to the input ports for another iteration of different phases. Note that the outputs are obtained from PE2 to PEM. The feedback is, however, directed to the processors from PE1 to PE$(M-1)$. Since $\boldsymbol{r}_2^{t^T}$ take the first $M-1$ elements of $\boldsymbol{r}_1^{b^T}$, it occupies the first $M-1$ processing cells. The second iteration is started once the fedback data are available. It is fully pipelined without any intermediate data arrangement and interrupt. The iterations are then continued until all the rows of R are obtained. The overall data arrangement is given in Fig. 4. Since only a left shift is performed in the feedback loop, it is obvious that a linear array as shown in Fig. 5 can be used without the need of feedback.

The number of time steps required for this linear array implementation is now being further reduced to $2(N-M) + (5(M-1)+1) = 2N + 3M - 4$ (or $2m + 5n - 4$ for a $2m \times n$ Toeplitz–Hankel matrix) which is linearly proportional to either M or N (m or n).

If the LS weight vector is of interest, another phase for the backward substitution can be started easily since all the data are now available in the linear array. The details of the operations of the backward substitution using a linear array can be found in [11].

V. CONCLUSIONS

In this correspondence, we propose a fast algorithm for the QRD of a Toeplitz–Hankel matrix. The computational complexity for the QRD of a $2m \times n$ Toeplitz–Hankel matrix is $10n^2 + 4mn + O(n)$ multiplications, which has an order of magnitude improvement over conventional algorithms. This algorithm can also be implemented onto a fully pipelined multiphase linear systolic array. The number of time steps required is further reduced to $2m + 5 - 4$ for the parallel implementation. An interesting point for the QRD of the specially structured matrices such as Toeplitz and Toeplitz–Hankel forms is that there is no need to store all the generated rows of the

TABLE II
THE OPERATIONS OF THE PROCESSING CELLS IN DIFFERENT PHASES

	Initialization	Phases 1 and 3	Phases 2, 4, and 5
PE1	$d = \sqrt{r^2 + x^2}$ $c = r/d,\ s = x/d$ $r = d$	$d = \sqrt{r^2 + x^2}$ $c = r/d,\ s = x/d$ $r = d$	$d = \sqrt{r^2 - x^2}$ $c = d/r,\ s = x/r$ $r = d$
PEi, $1 \leq i \leq M + 1$	$r = cr + sx$ $y = -sr + cx$	$r = cr + sx$ $y = -sr + cx$	$r = (r - sx)/c$ $y = -sr + cx$

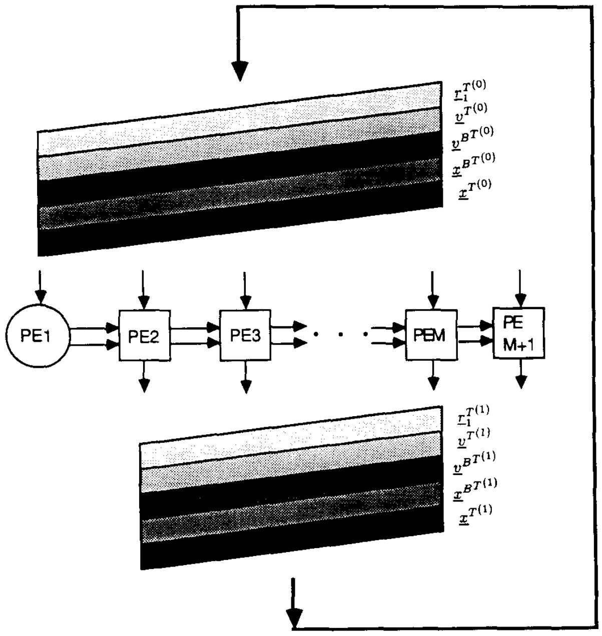

Fig. 3. The first iteration with five different phases.

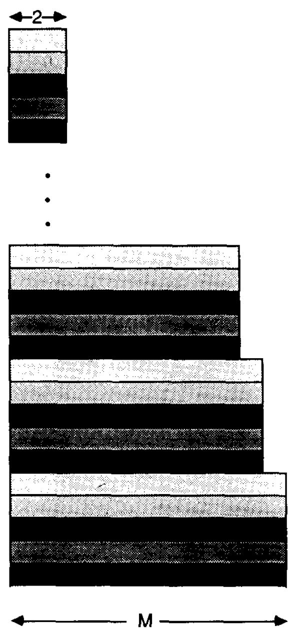

Fig. 4. The overall data arrangement.

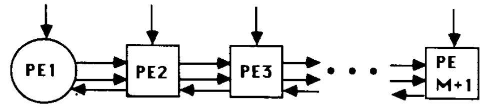

Fig. 5. The linear array for parallel implementations without feedback.

upper triangular matrix R. As long as the first row of R is known, all the subsequent row can be generated recursively, and this is also the basic principle of the proposed fast algorithm.

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The Relationship Between Instantaneous Frequency and Time-Frequency Representations

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***Abstract*—We give the relationship between instantaneous frequency estimation via the derivative of the phase of the analytic signal and the first moment of general time-frequency representations from Cohen's class in both the continuous and discrete-time domains. Many researchers have applied the standard linear definition of first moment to discrete-time time-frequency representations although this leads to biased instantaneous frequency estimators with high variance; we show that periodic (circular) definitions of moments must be used to account for the periodization of the frequency variable due to sampling.**

Manuscript received March, 2, 1990; revised January 25, 1992. This work was supported by the Australian Research Council.

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IEEE Log Number 9206002.

I. Introduction

Several authors [3], [9] have investigated the possibility of using the first moments of time-frequency representations with respect to the frequency variable as estimators of instantaneous frequency. This correspondence derives the relationships between instantaneous frequency and the first moments of the general class of time-frequency representations for both continuous and discrete-time signals.

II. Continuous-Time Estimation

Consider a frequency modulated sinusoidal signal of the form $x(t) = a_c(t) \cos \phi(t)$, where a_c represents the envelope function and ϕ is the cumulative phase of the signal. We define the instantaneous frequency of this signal by

$$f_i(t) = \frac{1}{2\pi} \frac{d\phi(t)}{dt}. \tag{1}$$

If x is sufficiently narrow band, a good estimate of the cumulative phase reduced modulo 2π may be obtained from the phase of the analytic signal defined as follows:

Definition 1: Analytic Signal: The analytic signal z associated with the real signal x is defined by $z = A[x]$, where $A[x] = x + jH[x]$ is the operator which forms the analytic signal and $H[\,]$ is the Hilbert transform defined by

$$H[x](t) = \frac{1}{\pi} \lim_{\delta \to 0} \left[\int_{-\infty}^{-\delta} \frac{x(t - \zeta)}{\zeta}\, d\zeta + \int_{\delta}^{+\infty} \frac{x(t - \zeta)}{\zeta}\, d\zeta \right].$$

We use the derivative of the phase of the analytic signal to define the following instantaneous frequency estimator.

Definition 2: Analytic Derivative Estimator: Let $z = A[x]$, where x is a real signal. Then the instantaneous frequency of x at time t is estimated by

$$\begin{aligned} \hat{f}_i^a(t) &= \frac{1}{2\pi} \left(\left(\frac{d}{dt} \right) \right)_{2\pi} \arg\ [z(t)] \\ &= \lim_{\delta \to 0} \frac{1}{4\pi\delta} ((\arg [z(t + \delta)] - \arg [z(t - \delta)]))_{2\pi} \end{aligned} \tag{2}$$

where $((\,))_{2\pi}$ denotes reduction modulo 2π and $((d/dt))_{2\pi}$ denotes the appropriate differentiation of a quantity which is reduced modulo 2π as shown above.

The spectrogram (or magnitude-squared short-time Fourier transform) and time-frequency distributions such as the Wigner–Ville, Born–Jordan–Cohen, Margenau–Hill–Rihaczek, and Choi–Williams exponential distributions can all be examined within the framework of Cohen's general class of time-frequency representations [2].

Definition 3: Cohen's Class of Time-Frequency Representations for Analytic Signals: Each member of this class of *bilinear* rep-